Initial data for binary black holes: the conformal thin-sandwich puncture method

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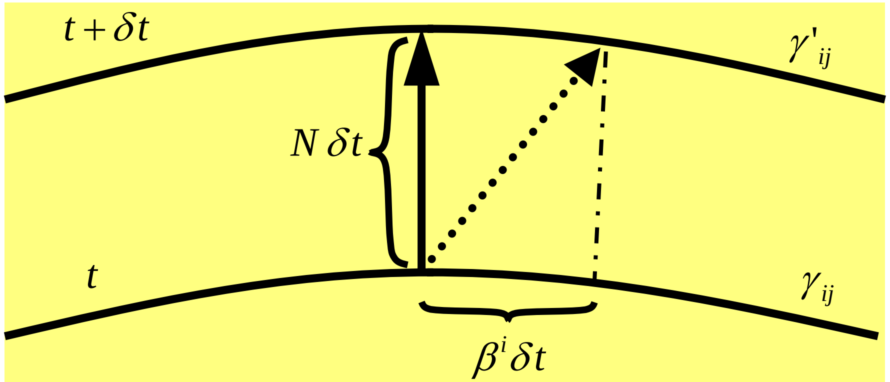
Overview: the smallest picture possible

- We want to simulate a (realistic) binary black hole collision. To do that,
    1. Rewrite Einstein's equations as a Cauchy problem
    2. Set up initial data for two black holes in orbit
    3. Evolve the system.
- Problems: we can't do (2) or (3) very well.
- Partial solution: try to create good initial data close to the interesting physics…
- Describe two black holes in quasi-circular, quasi-equilibrium orbit just before they plunge together.

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \qquad \mu,\nu = 0,1,2,3$$

Space and time are mixed…



Initial data: γ_{ij}, K_{ij}

$$K_{ij} = -\frac{1}{2N}(\partial_t \gamma_{ij} - \overline{\nabla}_i \beta_j - \overline{\nabla}_j \beta_i)$$

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\left.\begin{aligned} &\overline{R} + K^2 - K_{ij}K^{ij} = 16\pi G\rho \\ &\overline{\nabla}_l (K^{li} - \gamma^{li} K) = 8\pi G j^i \end{aligned}\right\}$$

Initial value constraints

$$\left.\begin{aligned} \partial_t K_{ij} = {} & N(\overline{R}_{ij} + KK_{ij} - 2K_{il}K^l_j) - \overline{\nabla}_i \overline{\nabla}_j N \\ & + \beta^l \overline{\nabla}_l K_{ij} + K_{il} \overline{\nabla}_j \beta^l + K_{jl} \overline{\nabla}_i \beta^l \\ & - 4\pi G N (2S_{ij} - \gamma_{ij}(S - \rho)) \\ \partial_t \gamma_{ij} = {} & -2NK_{ij} + \overline{\nabla}_i \beta_j + \overline{\nabla}_j \beta_i \end{aligned}\right\}$$

Evolution equations

What quantities are constrained?

$\gamma_{ij}, K_{ij} \quad \Rightarrow$ 12 independent components

- 4 constraint equations

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8 free quantities: 4 dynamical

4 gauge

Which are which?

Use a conformal decomposition…

Conformal thin-sandwich decomposition

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \qquad \beta^i = \tilde{\beta}^i \qquad N = \psi^6 \tilde{N}$$

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij} K \qquad A_{ij} = \psi^{-2}\tilde{A}_{ij} \qquad K = \tilde{K}$$

$$\tilde{A}_{ij} = \frac{1}{2\tilde{N}}\left[\left(\tilde{L}\beta\right)_{ij} - \tilde{u}_{ij}\right],$$

From $\partial_t K = \ldots$

$$\left(\tilde{L}\beta\right)_{ij} = \tilde{\nabla}_i \beta_j + \tilde{\nabla}_j \beta_i - \frac{2}{3}\tilde{\gamma}_{ij}\tilde{\nabla}_k \beta^k, \qquad \tilde{u}_{ij} = \partial_t \tilde{\gamma}_{ij}$$

$$\tilde{\nabla}^2 \psi - \tfrac{1}{8}\psi \tilde{R} - \tfrac{1}{8}\psi^5 K^2 + \tfrac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi G \psi^5 \rho$$

$$\tilde{\Delta}_L \beta^i - \left(\tilde{L}\beta\right)^{ij}\tilde{\nabla}_j \ln \tilde{N} - \frac{4}{3}\tilde{N}\psi^6 \tilde{\nabla}^i K = \tilde{N}\tilde{\nabla}_j\left(\frac{1}{\tilde{N}}\tilde{u}^{ij}\right) + 16\pi \tilde{N}\psi^{10} j^i$$

$$\tilde{\nabla}^2(\tilde{N}\psi^7) = (\tilde{N}\psi^7)\left[\frac{7}{8}\psi^{-8}\tilde{A}_{ij}\tilde{A}^{ij} + 2\pi G\psi^4(\rho + 2S) + \frac{5}{12}\psi^5 K^2\right] + \psi^5\left(\beta^l \tilde{\nabla}_l K - \boxed{\partial_t K}\right)$$

CTS: the essentials

- Free data: $\tilde{\gamma}_{ij}$ $\quad K$

  $\partial_t \tilde{\gamma}_{ij}$ $\quad \partial_t K$

- Solve for: $\psi, \ \beta^i, \ \tilde{N}$

- Construct: γ_{ij}, K_{ij}

"Easy" examples

- Schwarzschild (single stationary black hole):

$$\tilde{\gamma}_{ij} = f_{ij}, \quad \partial_t \tilde{\gamma}_{ij} = 0, \quad K = 0, \quad \partial_t K = 0$$

$$\psi = 1 + \frac{M}{2\tilde{r}}, \;\; \tilde{N}\psi^7 = 1 - \frac{M}{2\tilde{r}}, \quad \beta^i = 0 \quad \left(\tilde{A}_{ij} = 0\right)$$

- Brill-Lindquist (multiple stationary black holes)

$$\tilde{\gamma}_{ij} = f_{ij}, \quad \partial_t \tilde{\gamma}_{ij} = 0, \quad K = 0, \quad \partial_t K = 0$$

$$\psi = 1 + \sum_i^N \frac{m_i}{2\tilde{r}_i}, \;\; \beta^i = 0 \quad \left(\tilde{A}_{ij} = 0\right)$$

$$\tilde{N}\psi^7 = 1 + \sum_i^N \frac{c_i}{2\tilde{r}_i}, \qquad (c_i \text{ are undetermined...})$$

Orbits in the CTS decomposition

- In a corotating reference frame, the black holes will be almost stationary.
- Choose $\tilde{u}_{ij} = 0, \quad \partial_t K = 0$
- These choices are physically motivated
  - Free data choices in old decompositions were made for convenience

CTS solutions

- Gourgoulhon, Grandclément, and Bonazzola (GGB), 2001.
  - Solved with $\tilde{\gamma}_{ij} = f_{ij}, \quad K = 0$
  - Excised regions containing singularities
  - Employed boundary conditions on excised surfaces

(…there were inconsistencies here)

- I want to avoid inner boundary conditions

→ Puncture method.

CTS-puncture approach

Recall Brill-Lindquist solution:

$$\psi = 1 + \sum_i^N \frac{m_i}{2\tilde{r}_i} + u$$

Hamiltonian constraint:

$$\tilde{\nabla}^2 u = -\frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{12}\psi^5 K^2$$

Regular if $\tilde{A}_{ij}\tilde{A}^{ij} \sim \tilde{r}^{-6}$

$K \sim \tilde{r}^3$

Solve for u

Extend to $\tilde{N}\psi^7$:

$$\tilde{N}\psi^7 = 1 + \sum_i^N \frac{c_i}{2\tilde{r}_i} + v$$

(what are c_i?)

Constant-K equation:

$$\tilde{\nabla}^2 v = \left(1 + \sum_i^N \frac{c_i}{2\tilde{r}_i} + v\right)\left[\frac{7}{8}\psi^{-8}\tilde{A}_{ij}\tilde{A}^{ij} + \frac{5}{12}\psi^5 K^2\right] + \psi^5 \beta^l \tilde{\nabla}_l K$$

Solve for v

The shift has no singular part

- What corresponds to black holes with P^i and S^i?

Issues: Slicing choices (for one black hole)

$$\tilde{N}\psi^7 = 1 + \frac{c}{2\tilde{r}}$$

- Two principle choices: $c = \pm m$
- $c = -m$ Schwarzschild

$$\tilde{N}\psi^7 = 1 - \frac{m}{2\tilde{r}}$$

  – But: $\tilde{N} = 0$ on some surface…

$$\tilde{A}_{ij} = \frac{1}{2\tilde{N}}\left(\tilde{L}\beta\right)_{ij}$$

- $c = m$ Estabrook (N = 1).

$$\tilde{N}\psi^7 = 1 + \frac{m}{2\tilde{r}}$$

$\tilde{N} \geq 1$ but $\partial_t \tilde{A}_{ij} \neq 0$

  – This is a "dynamical" slicing!
  – The stationary Schwarzschild black hole will APPEAR to have dynamics

- **This isn't necessarily fatal to the method**

[Ref: MDH, C.R. Evans, G.B. Cook, T.W. Baumgarte, gr/qc-0306028]

Issue #2: The shift vector Conditions at the puncture?

- The analytic, singular part of the conformal factor gave us a black hole solution, without the need for inner boundary conditions
- There is no known analytic part of the shift for a black hole with non-zero P^i and S^i, and the puncture form of the lapse.
- We need to impose suitable conditions at the puncture.
- Methods to date do not give convergent results…

Future work

- Convert code to Cactus, where much greater resolution is possible
  - Maybe the momentum constraint solver will converge.
- Construct data with an everywhere positive lapse
- Examine the level of stationarity of quasi-circular orbits (located by, for example, the effective potential method)
  - Maybe the “Estabrook” lapse choice is Ok.