Mastodon Theorem - 20 Years in the Making

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Abstract

The *Mastodon theorem* (PD., D. Legg, D. Townsend, 2002), establishes that the regular bi-pyramid (North and South poles, and an equilateral triangle on the Equator) is the unique up to rotation five-point configuration on the sphere that maximizes the product of all mutual distances. More generally, given a configuration of points $\{x_1, \ldots, x_N\}$ on the unit sphere in $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$, its *Riesz s-energy* is defined as

$$\sum_{1 \leq i < j \leq N} \frac{1}{\|x_i - x_j\|^s}, \quad s > 0; \quad \sum_{1 \leq i < j \leq N} \log \frac{1}{\|x_i - x_j\|}, \quad s = 0.$$

The regular bi-pyramid minimizes the *logarithmic energy* ($s = 0$ case) for five points on $\mathbb{S}^2$.

Optimal point configurations that minimize the s-energy have broad applications in sciences, economics, information theory, etc. Rigorous proofs of optimality are extremely hard though. Even the important Coulomb energy ($s = 1$) case for five points on the unit sphere in 3-D space was resolved only recently (2013) by Richard Schwartz utilizing a computer-aided proof. In a subsequent monograph Schwartz extends the optimality of the bipyramid to all $s < s^*$.

In a joint work with Oleg Musin we generalize the Mastodon Theorem to $n + 2$ points on $\mathbb{S}^{n-1}$, namely we characterize all stationary configurations, and show that all local minima occur when a configuration splits in two orthogonal simplexes of k and ℓ vertices, $k + \ell = n + 2$, with global minimum attained when $k = \ell$ or $k = \ell + 1$ depending on the parity of n.

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