Universal lower bounds for the energy of spherical codes: lifting the Levenshtein framework.

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Abstract

For a potential h defined on the interval $[-1, 1)$ and a finite point configuration (or *code*) C on the unit sphere $\mathbb{S}^{n-1}$, the h-energy of C is given by

$$E(C, n, h) := \sum_{x \neq y \in C} h(\langle x, y \rangle).$$

H. Cohn and A. Kumar proved that every sharp spherical code (a code with m distinct inner products that is also a spherical design of strength $2m-1$) is universally optimal; i.e., such a code minimizes the potential h-energy among all codes of cardinality $|C|$ for any absolutely monotone potential h. In addition, they proved that the 600-cell (a code with 120 points on $\mathbb{S}^3$) is also a universally optimal code.

In this talk we present lower bounds for energy of the form

$$E(C, n, h) \geq N^2 \sum_{i=1}^{m} \rho_i h(\alpha_i),$$

where the nodes $\{\alpha_i\}$ and weights $\{\rho_i\}$ depend only on the cardinality N and dimension n and are obtained from a quadrature rule framework developed by Levenshtein in relation to maximal codes. The Hermite interpolation polynomials to h at $\{\alpha_i\}$ solve a Delsarte-Yudin linear program for polynomials of degree at most $2m-2$ or $2m-1$ depending on whether -1 is among the nodes. We extend these bounds by considering the linear program for polynomials of higher degree, essentially lifting the Levenshtein framework and apply these techniques to obtain universal lower bounds for the interesting cases of $N = 24$ and 120 points on $\mathbb{S}^3$. In the case $N = 24$ we obtain the best possible linear programming bound and in the case $N = 120$ we obtain a simplified and (somewhat) systematic proof of the universal optimality of the 600-cell.