## Fans of faces of parallelohedral tilings Alexander Magazinov (Steklov Mathematical Institute)

## Abstract

A parallelohedron is a convex polytope P that admits a face-to-face tiling  $T_P$  of  $\mathbb{R}^d$  by its translates.

The central conjecture concerning parallelohedra is the one by G. Voronoi (see [2]).

**Conjecture 1.** Every d-dimensional parallelohedron P is affinely equivalent to a Dirichlet-Voronoi domain for some d-dimensional lattice.

Conjecture 1 has not been proved or disproved so far in full generality. However, several significant partial results have been obtained. For many approaches the study of local structure of  $T_P$  is important.

**Definition 1.** Denote by  $\pi$  the projection along lin F onto the complementary affine space  $(\lim F)^{\text{compl}}$ . Then there exists a complete k-dimensional poyhedral fan fan(F) (the fan of F) that splits  $(\lim F)^{\text{compl}}$  into convex polyhedral cones with vertex  $\pi(F)$ , and a neighborhood  $U = U(\pi(F))$  such that every face  $F' \supset F$  corresponds to a cone  $C \in \text{fan}(F)$  satisfying

$$\pi(F') \cap U = C \cap U.$$

*Remark* 1. Speaking informally, fan(F) has the same combinatorial structure as the transversal section of  $T_P$  in a small neighborhood of F.

In 1929 B.N. Delaunay (see [1]) proved the key result.

**Theorem 1.** Let P be a d-dimensional parallelohedron and F be a (d-3)imensional face of  $T_P$ . Then fan(F) has one of the 5 combinatorial types shown in Figure 1. Moreover, each of these types is realized for some 3imensional tiling.

We give a combinatorial proof of Theorem 1. Also we prove a general result on fans of faces.

Definition 2. Let

$$\nu(F) = card \{ P' \in \mathcal{T}(P) : F \subset P' \}.$$

 $\nu(F)$  is called the *valence* of the face F.



Figure 1: 5 possible fans of (d-3)-faces

**Theorem 2.** Let P be a d-dimensional parallelohedron and F be a (d-k)imensional face of  $T_P$ . Then

$$\nu(F) \le 2^k.$$

Theorem 2 immediately implies

**Corollary 3.** Given  $k \in \mathbb{N}$ , there exists a set of complete k-dimensional polyhedral fans

$$\{\mathcal{C}_1^k, \mathcal{C}_2^k, \dots, \mathcal{C}_{N(k)}^k\}$$

such that for every d, every d-parallelohedron P and every (d-k)-face F of  $T_P$  the fan of F is isomorphic to some  $C_i^k$ .

## References

- B. N. Delaunay, Sur la partition régulière de l'espace à 4 dimensions, Izv. Acad. sci. of the USSR. Ser. VII. Sect. of phys. and math. sci., 1 – 2 (1929), 79 – 110, 147 – 164.
- [2] G. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Deuxième mémoire. Recherches sur les paralléloèdres primitifs, J. Reine Angew. Math., 134 (1908), 198 – 287 and 136 (1909), 67 – 178.