A characterization of great antipodal sets of complex Grassmannian manifolds by designs

Hirotake Kurihara (Kitakyushu National College of Technology)
Email: kurihara@kurims.kyoto-u.ac.jp

Abstract

This is joint work with Takayuki Okuda. In this talk, we give a characterization of great antipodal sets of complex Grassmannian manifolds as certain designs with the smallest cardinality.

We first give design theory on complex Grassmannian manifolds. Let $\mathcal{G}_{m,n}$ be the set of m-dimensional subspaces of $\mathbb{C}^n$, and $\mathcal{G}_{m,n}$ is called the *complex Grassmannian manifold*. It is well known that $\mathcal{G}_{m,n}$ is a symmetric space which is isomorphic to $U(n)/(U(m)\times U(n-m))$. By the highest weight theory, a complex irreducible unitary representation of $U(n)$ is determined by its highest weight, and the set of the irreducible representations of $U(n)$, up to isomorphisms, can be regarded as the set of n-tuples of integers. Let us denote by $C^0(\mathcal{G}_{m,n})$ the functional space consisted of all $\mathbb{C}$-valued continuous functions on $\mathcal{G}_{m,n}$. The index of an irreducible representation H_μ of $U(n)$ in $C^0(\mathcal{G}_{m,n})$ must be of the form of $(\mu, 0, \ldots, 0, -\mu)$, where $\mu \in P_m := \{\mu = (\mu_1, \ldots, \mu_m) \mid \mu_1 \geq \cdots \geq \mu_m \geq 0,\ \mu_i \in \mathbb{Z}\}$, and $\bigoplus_{\mu \in P_m} H_\mu$ is dense in $C^0(\mathcal{G}_{m,n})$. Let $\mathcal{T}$ be a finite subset of P_m. Then a nonempty finite subset X of $\mathcal{G}_{m,n}$ is called a $\mathcal{T}$*-design* if for any $f \in \bigoplus_{\mu\in\mathcal{T}} H_\mu$,

$$\frac{1}{\nu(\mathcal{G}_{m,n})}\int_{\mathcal{G}_{m,n}} f d\nu = \frac{1}{|X|}\sum_{\mathfrak{a}\in X} f(\mathfrak{a})$$

holds, where ν is a $U(n)$-invariant Haar measure on $\mathcal{G}_{m,n}$.

In the case of spherical designs, tight spherical odd designs relate to antipodal pairs. In particular, X is a tight spherical 1-design if and only if X is an antipodal pair $\{x, -x\}$. We second give the definition and some properties of great antipodal sets on $\mathcal{G}_{m,n}$. For $\mathfrak{a} \in \mathcal{G}_{m,n}$, we denote by $s_\mathfrak{a}$ the point symmetry at $\mathfrak{a}$. A subset S of $\mathcal{G}_{m,n}$ is called an *antipodal set* if $s_\mathfrak{a}(\mathfrak{b}) = \mathfrak{b}$ for any $\mathfrak{a}, \mathfrak{b} \in S$. An antipodal set is a generalization of an antipodal pair on spheres. An antipodal set S is called *great* if $|S| = \max\{|S'| \mid S' \text{ is an antipodal set}\}$. Then it is known that $S = \{\mathrm{Span}\{e_{i_k}\}_{k=1}^m | 1 \leq i_1 < i_2 < \cdots < i_m \leq n\}$ is a great antipodal set of $\mathcal{G}_{m,n}$, where $\{e_i\}_{i=1}^n$ is an orthonormal basis of $\mathbb{C}^n$.

In $\mathcal{G}_{m,n}$, we can also characterize great antipodal sets by certain tight designs. Let $\mathcal{E} := \{(1^i)\}_{i=1}^m$ and $\mathcal{F} := \{(2, 1^{i-1})\}_{i=2}^m$ in P_m. The following are our results.

1. A great antipodal set S is a $\mathcal{E} \cup \mathcal{F}$-design.
2. Let X be a $\mathcal{E} \cup \mathcal{F}$-design on $\mathcal{G}_{m,n}$. Then $|X| \geq \binom{n}{m}$ $(= |S|)$ holds.
3. Let X be a subset of $\mathcal{G}_{m,n}$ with $|X| = \binom{n}{m}$. Then the following are equivalent: (a) X is a $\mathcal{E} \cup \mathcal{F}$-design, (b) X is a great antipodal set.

The last result yields a characterization of great antipodal sets by using design theory.