Problems and results of the Buchstaber invariant theory for simple polytopes and simplicial complexes

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Abstract

The Buchstaber invariant is a combinatorial invariant of simple polytopes and simplicial complexes that comes from toric topology [BP]. With each simplicial $(n-1)$-dimensional complex K on m vertices we can associate a topological space – $(m+n)$-dimensional moment-angle complex $\mathcal{Z}_K$ with a canonical action of a compact torus T^m. The topology of $\mathcal{Z}_K$ and of the action depends only on the combinatorics of K, which gives a tool to study the combinatorics of polytopes and simplicial complexes in terms of the algebraic topology of moment-angle complexes and vice versa. A Buchstaber invariant $s(K)$ is equal to the maximal dimension of torus subgroups $H \subset T^m$, $H \simeq T^k$, that act freely on $\mathcal{Z}_K$. If K is not the full simplex, then $1 \leqslant s(K) \leqslant m-n$.

The Buchstaber invariant has been studied since 2001 by I. Izmestiev, M. Masuda and Y. Fukukawa, A. Ayzenberg, the author [E1, E2], and some others.

The following questions are in the focus of the theory:

1. *To find an effective combinatorial description of* $s(K)$*.* There are two dual combinatorial descriptions corresponding to two descriptions of a torus subgroup: parametrically and as a kernel of a map; and the description in terms of the set $N(K)$ of minimal non-faces. For a polytope P we have $s(P)=1$ iff P is a simplex. The case of polytopes with $s(P)=2$ is much wider. We will show the criterion for $s(K)=2$ in terms of $N(K)$.

2. *To describe* $s(K)$ *for different classes of polytopes and complexes.* It is known that for 3-polytopes $s(P)=m-3$. A. Ayzenberg showed that $s(\Gamma)=m-\lceil\log_2(\gamma(\Gamma)+1)\rceil$ for a graph Γ. For an n-polytope P with $m=n+3$ we have $s(P)=3$ iff $|N(P)|\leqslant 7$. M. Masuda and Y. Fukukawa obtained nontrivial results in the important case of $K=\Delta^{m-1}_{n-1}$ – an $(n-1)$-skeleton of an $(m-1)$-simplex.

3. *To describe the connection with other classical and modern combinatorial invariants such as the chromatic number* $\gamma(K)$ *and the bigraded Betti numbers.* A. Ayzenberg noted that $s(K)$ can be considered as a generalization of $\gamma(K)$. It is known that $s(P)$ can not be calculated if only the

f-vector of P is given. The natural question is to find the connection with the bigraded Betti numbers of K.

4. *To find lower and upper bounds for the Buchstaber invariant.* Now we know that

$$\left[\frac{m}{n+1}\right] \leqslant m-\gamma(K)+s(\Delta_{n-1}^{\gamma-1}) \leqslant s(K) \leqslant m-\lceil\log_2(\gamma(K)+1)\rceil.$$

5. *To describe the behaviour under constructions and operations on polytopes and complexes.* For example, $|s(P)-s(P')| \leqslant 1$ if the polytope P' is obtained from P by an i-flip, $2 \leqslant i \leqslant n-1$, and $s(P)+1 \leqslant s(P') \leqslant s(P)+2$, if P' is obtained from P by cutting off a vertex. We have

$$s(P)+s(Q) \leqslant s(P\times Q) \leqslant s(P)+s(Q)+\min\{m_1-n_1-s(P), m_2-n_2-s(Q)\}.$$

References

[BP] V. M. Buchstaber, T. E. Panov, *Torus actions and their applications in topology and combinatorics*, Providence, R.I.: American Mathematical Society, 2002. (University Lecture Series; V.24).

[E1] N. Erokhovets, *Buchstaber Invariant of Simple Polytopes*, arXiv: 0908.3407.

[E2] N. Erokhovets, *Criterion for the Buchstaber invariant of simplicial complexes to be equal to two*, arXiv:1212.3970.