Schedule of talks.

Thursday (April 18)
Morning session

9:00-9:40

Francis Edward Su (Harvey Mudd College)

Combinatorial fixed point theorems

Abstract. The Brouwer fixed point theorem and the Borsuk-Ulam theorem are beautiful and well-known theorems of topology. It is perhaps less well-known that the Borsuk-Ulam theorem implies the Brouwer fixed point theorem, and that these theorems both admit combinatorial analogues. In particular, Sperner's lemma is equivalent to the Brouwer fixed point theorem, and Tucker's lemma is equivalent of the Borsuk-Ulam theorem. With these theorems, I will trace recent connections, applications, and generalizations—some of which includes research with undergraduates.

Thursday (April 18)
Morning session

9:45-10:25

Marjorie Senechal (Smith College)

Crystals and nanoclusters: a geometry puzzle?

Abstract. As B. N. Delone once said, "Tradition ascribes to Plato the discovery of the five regular convex solids . . . and Fedorov discovered the five parallelohedra." Indeed, for a century after Fedorov's 1885 discovery, the parallelohedra were the more momentous for crystallography, because they characterize periodic crystal structures. A century after Fedorov, the discovery of quasicrystals overthrew the periodicity paradigm. Now crystallographers are turning to Plato again, modeling condensed matter not by tilings but by dense packings and coverings of tetrahedral and icosahedral nanoclusters. I will outline the geometry questions this new viewpoint poses.

Thursday (April 18)
Morning session

10:50-11:30

Nikolay Erokhovets (Moscow State University & Delone Laboratory of Yaroslavl State University)

Problems and results of the Buchstaber invariant theory for simple polytopes and simplicial complexes

Abstract. The Buchstaber invariant is a combinatorial invariant of simple polytopes and simplicial complexes that comes from toric topology [BP]. With each simplicial $(n-1)$-dimensional complex K on m vertices we can associate a topological space – $(m+n)$-dimensional moment-angle complex $\mathcal{Z}_K$ with a canonical action of a compact torus T^m. The topology of $\mathcal{Z}_K$ and of the action depends only on the combinatorics of K, which gives a tool to study the combinatorics of polytopes and simplicial complexes in terms of the algebraic topology of moment-angle complexes and vice versa. A Buchstaber invariant $s(K)$ is equal to the maximal dimension of torus subgroups $H \subset T^m$, $H \simeq T^k$, that act freely on $\mathcal{Z}_K$. If K is not the full simplex, then $1 \leqslant s(K) \leqslant m-n$.

The Buchstaber invariant has been studied since 2001 by I. Izmestiev, M. Masuda and Y. Fukukawa, A. Ayzenberg, the author [E1, E2], and some others.

The following questions are in the focus of the theory:

1. *To find an effective combinatorial description of* $s(K)$. There are two dual combinatorial descriptions corresponding to two descriptions of a torus subgroup: parametrically and as a kernel of a map; and the description in terms of the set $N(K)$ of minimal non-faces. For a polytope P we have $s(P) = 1$ iff P is a simplex. The case of polytopes with $s(P) = 2$ is much wider. We will show the criterion for $s(K) = 2$ in terms of $N(K)$.

2. *To describe $s(K)$ for different classes of polytopes and complexes.* It is known that for 3-polytopes $s(P) = m - 3$. A. Ayzenberg showed that $s(\Gamma) = m - \lceil \log_2(\gamma(\Gamma) + 1) \rceil$ for a graph Γ. For an n-polytope P with $m = n + 3$ we have $s(P) = 3$ iff $|N(P)| \leqslant 7$. M. Masuda and Y. Fukukawa obtained nontrivial results in the important case of $K = \Delta_{n-1}^{m-1}$ – an $(n-1)$-skeleton of an $(m-1)$-simplex.

3. *To describe the connection with other classical and modern combinatorial invariants such as the chromatic number $\gamma(K)$ and the bigraded Betti numbers.* A. Ayzenberg noted that $s(K)$ can be considered as a generalization of $\gamma(K)$. It is known that $s(P)$ can not be calculated if only the f-vector of P is given. The natural question is to find the connection with the bigraded Betti numbers of K.

4. *To find lower and upper bounds for the Buchstaber invariant.* Now we know that

$$\left[\frac{m}{n+1}\right] \leqslant m - \gamma(K) + s(\Delta_{n-1}^{\gamma-1}) \leqslant s(K) \leqslant m - \lceil \log_2(\gamma(K) + 1) \rceil.$$

5. *To describe the behaviour under constructions and operations on polytopes and complexes.* For example, $|s(P) - s(P')| \leqslant 1$ if the polytope P' is obtained from P by an i-flip, $2 \leqslant i \leqslant n - 1$, and $s(P) + 1 \leqslant s(P') \leqslant s(P) + 2$, if P' is obtained from P by cutting off a vertex. We have

$$s(P)+s(Q) \leqslant s(P\times Q) \leqslant s(P)+s(Q)+\min\{m_1-n_1-s(P), m_2-n_2-s(Q)\}.$$

References

[BP] V. M. Buchstaber, T. E. Panov, *Torus actions and their applications in topology and combinatorics*, Providence, R.I.: American Mathematical Society, 2002. (University Lecture Series; V.24).

[E1] N. Erokhovets, *Buchstaber Invariant of Simple Polytopes*, arXiv: 0908.3407.

[E2] N. Erokhovets, *Criterion for the Buchstaber invariant of simplicial complexes to be equal to two*, arXiv:1212.3970.

Thursday (April 18)
Morning session

11:35-12:15

Alexey Garber (Moscow State University; Delone Lab, Yaroslavl)

with Dirk Frettlöh (Bielefeld University)

Tilings with unique vertex corona

Abstract. The well-known Local Theorem by N. Dolbilin establishes local conditions on tile neighbourhoods of a tiling $\mathcal{T}$ that are necessary and sufficient for $\mathcal{T}$ being crystallographic.

Theorem 1 (N. Dolbilin)**.** *A tiling $\mathcal{T}$ in d-dimensional Euclidean of hyperbolic space is crystallogrpaphic iff the following two conditions hold for some $k \geq 0$:*

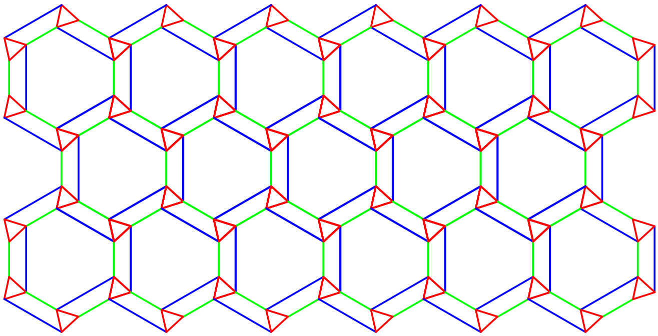
(1) *For the numbers $N(k)$ of k-coronae (i.e. union of all tiles in the k-th surrounding of a given tile) holds: $N(k+1) = N(k)$, and $N(k)$ is finite.*

(2) *$S_i(k+1) = S_i(k)$ for $1 \leq i \leq N(k)$,*

where $S_i(k)$ denotes the symmetry group of the i-th k-corona.

The question arises whether there are local conditions on the vertex neighbourhoods that imply that the tiling is crystallographic. In particular, if all vertex coronas are congruent, is the tiling necessarily crystallographic?

In this talk we will present several possible families of tilings with unique vertex corona and post discuss which translation group such a tiling can have.



Thursday (April 18)
Afternoon session

2:00-2:40

A. O. Ivanov (Moscow State University)

with V. N. Sal'nikov

Probabilities of Optimal Networks Structures

Abstract. Steiner problem of finding a shortest tree connecting a given finite subset M of a metric space is known as NP-complete, see [1], and the main reason is an exponentially large number of possible structures of the trees which can connect the terminal set M and possible additional vertices-forks. Similar difficulties appear in other one-dimensional geometrical variational problems, such as minimal fillings problem [2], etc.

We suggest to investigate probabilistic characteristics of possible combinatorial structures of optimal networks, i.eẇe want to find out which structures appear quite often, and which — rather rarely.

As a main example we take minimal fillings of finite additive metric spaces, see definitions below. For that case a probabilistic model is constructed. Under this model, for a pair of binary trees with n vertices of degree 1 a formula characterizing the ratio of probabilities of these trees appearance as minimal filling structures is obtained [4]. An asymptotic behavior of the probabilities under $n \to \infty$ is investigated.

Necessary definitions. A connected weighted graph $\bigl(G = (V, E), \omega\bigr)$ with vertex set V, edge set E, and non-negative weight function $\omega\colon E \to \mathbb{R}$ is a *filling* of a finite metric space (M, ρ), see [2], if $M \subset V$ and for any points x, y from M the inequality $\rho(x, y) \le d_\omega(x, y)$ is valid, where d_ω is a distance function generated by ω on M. Namely, $d_\omega(x, y)$ is equal to the least possible weight of the pass connecting

x and y in G. A filling with the least possible weight is referred as *minimal filling*. Also recall, that a finite metric space (M, ρ) is said to be *additive*, if its distance function ρ can be generated by a *generating tree*, a weighted tree (T, ω), whose degree 1 vertex set coincides with M, and such that $\rho(x, y) = d_\omega(x, y)$ for any $x, y \in M$, see [3]. In [2] it is proved that minimal fillings of finite additive metric spaces are exactly their generating trees.

References

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Thursday (April 18)
Afternoon session

2:45-3:25

Filip Morić (École Polytechnique Fédérale de Lausanne)

with Radoslav Fulek, Yoshio Okamoto, Tibor Szabó and Csaba D. Tóth

Planar straight-line graphs with free edge lengths

Abstract. We study straight-line embeddings of planar graphs subject to metric constraints. A planar graph G is *free* in a planar "host" graph H, $G \subseteq H$, if the edges of G have arbitrary positive lengths, that is, for any choice of positive lengths for the edges of G, the host H has a straight-line embedding that realizes these lengths. A planar graph G is *extrinsically free* in H, $G \subseteq H$, if any constraint on the edge lengths of G depends on G alone, irrespective of any additional edges of the host H.

We characterize all planar graphs G that are free in any host H, $G \subseteq H$. We also give an almost complete characterization of the planar graphs G that are extrinsically free in any host H, $G \subseteq H$; the status of the cycles C_k, $k \geq 5$, remains open, leading to a new variant of the celebrated carpenter's rule problem. Separating triangles, and separating cycles in general, play an important role in our arguments. We show that every star is free in a 4-connected triangulation, which has no separating triangle.

Thursday (April 18)
Afternoon session

3:50-4:30

Gregory Minton (Massachusetts Institute of Technology)

Optimal codes in projective spaces

Abstract. It is known that, in projective spaces, a regular simplex which is also a 1-design is an optimal code. In fact, such a simplex is universally optimal. However, as the space and the number of points vary, these codes do not always exist. In real and complex projective spaces they seem to only exist in special cases, but in projective spaces over the quaternions and octonions their existence seems to be more common. In fact, simple dimension-counting arguments suggest that they should exist for a range of cardinalities, and moreover they should exist in positive-dimensional families. We show that these arguments can be formalized by computer-assisted proof; these proofs demonstrate rigorously that a true solution exists in a small neighborhood of a given approximate solution. The crux of our method is finding a minimal set of equalities determining a regular simplicial 1-design. Using this approach we prove the existence of a laundry list of new universally optimal codes, including tight 2-designs in the quaternionic and octonionic projective planes.

Thursday (April 18)
Afternoon session

4:35-4:55

Alexey Eremin (Moscow State University)

Minimal fillings and minimal metric hulls of infinite metric spaces

Abstract. The problem concerning minimal fillings of finite metric spaces was posed by Ivanov and Tuzhilin. The objective is to find a weighted graph of minimum weight among all weighted graphs joining the points of a given finite metric space provided that for any two points in the metric space the distance between them is not greater than the weight of the shortest path connecting them in the graph. We discuss possible generalizations of this problem to the case of infinite metric spaces. In the first part of this talk we discuss properties of existing notion of minimal filling in the case of infinite metric space and present some limit theorems. In the second part we introduce new notion of minimal metric hull of metric space (which in some natural sense generalizes notion of minimal filling) and discuss its properties and question arising.

Friday (April 19)
Morning session

9:00-9:40

Peter Dragnev (Indiana University – Purdue University Fort Wayne)

Characterizing stationary logarithmic points

Abstract. The product of all $N(N-1)/2$ possible distances for a collection of N points on the circle is maximized when the points are (up to rotation) the N-th roots of unity. There is an elegant elementary proof of this fact. In higher dimensions the problem becomes much more complicated. For example, if the points are restricted to the unit sphere in 3-space, the result is known for $N = 1 – 6$, and 12. We will derive a characterization theorem for the stationary points in d-space and illustrate it with a couple of examples of optimal configurations that are new in the literature.

Friday (April 19)
Morning session

9:45-10:25

Robert Erdahl (Queens University, Canada)

Commensurate Vectors, Commensurate Triangles and the Proliferation of Combinatorial Types of Parallelohedra

Abstract. One of the more striking, indeed daunting discoveries in the theory of parallelohedra is the astounding rate of growth of combinatorial types with dimension. For example, by restricting attention to the primitive types where the dual cell corresponding to each vertex is a simplex, this growth has been reported by Peter Engel to be:

Dimension	2	3	4	5	6
No. of types	1	1	3	222	185×10^6

The data available for the general case is much more forbidding. An important problem in the theory of parallelohedra is to gain some insight into how such rapid growth is possible.

In my talk I will describe a few simple construction that explain in part how rapid growth can occur in higher dimension, and even be expected. The constructions have as starting point the simple notions of commensurate lattice vector and commensurate lattice triangle. A commensurate lattice vector in the ambient lattice is one that can be translated so that it fits inside the parallelohedron, and similarly, a commensurate lattice triangle is one that can be translated so that it fits inside the parallelohedron.

Friday (April 19)
Morning session

10:50-11:30

Hiroshi Nozaki (Aichi University of Education)

Eigenvalues of large distance sets and its applications

Abstract. A finite subset X of the Euclidean space is called an s-distance set if the number of distances between distinct vectors of X is equal to s. We obtain a graph which has s relations from an s-distance set by a natural way. We show that if the size of an s-distance set is greater than some value then an eigenvalue of the graph becomes some special value, so called the generalized Larman-Rogers-Seidels ratio. By this result, we give some result about a Euclidean representation of a graph having s relations, and we also prove a new lower bound for the maximum distance of integral point sets. Here if an s-distance set satisfies all distances are integers, then it is called an integral point set.

Friday (April 19)
Morning session

11:35-12:15

Hirotake Kurihara (Kitakyushu National College of Technology)

with Takayuki Okuda (Tohoku University)

A characterization of great antipodal sets of complex Grassmannian manifolds by designs

Abstract. In this talk, we give a characterization of great antipodal sets of complex Grassmannian manifolds as certain designs with the smallest cardinality.

We first give design theory on complex Grassmannian manifolds. Let $\mathcal{G}_{m,n}$ be the set of m-dimensional subspaces of $\mathbb{C}^n$, and $\mathcal{G}_{m,n}$ is called the *complex Grassmannian manifold*. It is well known that $\mathcal{G}_{m,n}$ is a symmetric space which is isomorphic to $U(n)/(U(m) \times U(n-m))$. By the highest weight theory, a complex irreducible unitary representation of $U(n)$ is determined by its highest weight, and the set of the irreducible representations of $U(n)$, up to isomorphisms, can be regarded as the set of n-tuples of integers. Let us denote by $C^0(\mathcal{G}_{m,n})$ the functional space consisted of all $\mathbb{C}$-valued continuous functions on $\mathcal{G}_{m,n}$. The index of an irreducible representation H_μ of $U(n)$ in $C^0(\mathcal{G}_{m,n})$ must be of the form of $(\mu, 0, \ldots, 0, -\mu)$, where $\mu \in P_m := \{\mu = (\mu_1, \ldots, \mu_m) \mid \mu_1 \geq \cdots \geq \mu_m \geq 0,\ \mu_i \in \mathbb{Z}\}$, and $\bigoplus_{\mu \in P_m} H_\mu$ is dense in $C^0(\mathcal{G}_{m,n})$. Let $\mathcal{T}$ be a finite subset of P_m. Then a nonempty finite subset X of $\mathcal{G}_{m,n}$ is called a $\mathcal{T}$-*design* if for any $f \in \bigoplus_{\mu \in \mathcal{T}} H_\mu$,

$$\frac{1}{\nu(\mathcal{G}_{m,n})} \int_{\mathcal{G}_{m,n}} f d\nu = \frac{1}{|X|} \sum_{\mathfrak{a} \in X} f(\mathfrak{a})$$

holds, where ν is a $U(n)$-invariant Haar measure on $\mathcal{G}_{m,n}$.

In the case of spherical designs, tight spherical odd designs relate to antipodal pairs. In particular, X is a tight spherical 1-design if and only if X is an antipodal pair $\{x, -x\}$. We second give the definition and some properties of great antipodal sets on $\mathcal{G}_{m,n}$. For $\mathfrak{a} \in \mathcal{G}_{m,n}$, we denote by $s_{\mathfrak{a}}$ the point symmetry at $\mathfrak{a}$. A subset S of $\mathcal{G}_{m,n}$ is called an *antipodal set* if $s_{\mathfrak{a}}(\mathfrak{b}) = \mathfrak{b}$ for any $\mathfrak{a}, \mathfrak{b} \in S$. An antipodal set is a generalization of an antipodal pair on spheres. An antipodal set S is called *great* if $|S| = \max\{|S'| \mid S' \text{ is an antipodal set}\}$. Then it is known that $S = \{\mathrm{Span}\{e_{i_k}\}_{k=1}^m | 1 \leq i_1 < i_2 < \cdots < i_m \leq n\}$ is a great antipodal set of $\mathcal{G}_{m,n}$, where $\{e_i\}_{i=1}^n$ is an orthonormal basis of $\mathbb{C}^n$.

In $\mathcal{G}_{m,n}$, we can also characterize great antipodal sets by certain tight designs. Let $\mathcal{E} := \{(1^i)\}_{i=1}^m$ and $\mathcal{F} := \{(2, 1^{i-1})\}_{i=2}^m$ in P_m. The following are our results.

1. A great antipodal set S is a $\mathcal{E} \cup \mathcal{F}$-design.
2. Let X be a $\mathcal{E} \cup \mathcal{F}$-design on $\mathcal{G}_{m,n}$. Then $|X| \geq \binom{n}{m}$ $(= |S|)$ holds.
3. Let X be a subset of $\mathcal{G}_{m,n}$ with $|X| = \binom{n}{m}$. Then the following are equivalent: (a) X is a $\mathcal{E} \cup \mathcal{F}$-design, (b) X is a great antipodal set.

The last result yields a characterization of great antipodal sets by using design theory.

Friday (April 19)
Afternoon session

2:00-2:40

Ilya Dumer (University of California Riverside)

with Olga Kapralova (University of California Riverside)

Minimum weights of Boolean polynomials on the spherical Hamming layers

Abstract. Consider m-variate Boolean polynomials of degree r or less. Our goal is to find the minimum Hamming weights that these polynomials take on the sets of binary m-tuples of a given Hamming weight b. From the coding perspective, this setting defines a punctured binary Reed-Muller code $RM(r, m)$ whose positions form a Hamming sphere of weight b in the m-dimensional binary space. In this talk, we specify some recursive properties of this single-layer spherical construction $RM(r, m, b)$ and define its code parameters for any values of the input parameters m, r, and b. We also describe coding techniques that increase minimum distances of codes $RM(r, m, b)$ and obtain codes that meet the upper Griesmer bound.

Friday (April 19)
Afternoon session

2:45-3:25

Masanori Sawa (Nagoya University)

On Hilbert identities and designs on the simplex

Abstract. A Hilbert identity is a representation of $(x_1^2+\cdots+x_n^2)^r$ as a sum of $2r$-th powers of real linear forms $\alpha_1 x_1 + \cdots + \alpha_n x_n$, which originally stems from Hilbert's solution of Waring's Problem in number theory. There is a beautiful connection with cubature formulas on spheres. In this talk I will introduce this connection and related facts, and show a new relation between spherical cubature and "simplical" cubature. I will discuss what we know when translating spherical cubature into identities, or conversely, translating identities with cubature terminology. You can also enjoy many collaborators who appear in my talk!

Friday (April 19)
Afternoon session

3:50-4:30

Takayuki Okuda (Tohoku University)

A new construction of spherical designs by using Hopf maps

Abstract. It is known that one can make spherical t-designs on a d-sphere S^d from a spherical t-design on S^{d-1} and an interval t-design on the open interval $(-1, 1)$ with respect to the weight function $w_d(s) := (1 - s^2)^{(d-2)/2}$ ([Rabau–Bajnok, J. Approx. Theory (1991)], [Wagner, Monatsh. Math. (1991)]).

In this talk, we generalize the fact above and applying it for Hopf maps, then we have an algorithm to making spherical designs on S^3 [resp. S^7] from spherical designs on S^2 and S^1 [resp. S^4 and S^3].

Friday (April 19)
Afternoon session

4:35-5:15

Anton Nikitenko (Saint Petersburg State University, Chebyshev Lab, Delone Lab)

with Oleg R. Musin (University of Texas at Brownsville)

Optimal packings of disks on torus

Abstract. We consider packings of congruent circles on a square flat torus, i.e., periodic (w.r.t. a square lattice) planar circle packings, with the maximal circle radius. Similar problems of packing disks on different flat tori have been studied previously by R. Connelly ([1, 2]) and W. Dickinson ([3]); the particular case of the square torus is especially interesting due to a practical reason: D. Usikov has shown it to be important for the problem of ”super resolution of images” ([8]). In our work([5]) we have adopted an algorithm used to find optimal arrangements of spherical caps on a sphere ([6]) for the flat case and with its help have found optimal arrangements for six, seven and eight circles (optimal packings of up to five circles have been previously determined by Dickinson et al. in [4]).

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Saturday (April 20)
Morning session

9:00-9:40

Alexander Barvinok (University of Michigan)

Thrifty approximations of convex bodies by polytopes

Abstract. Given a d-dimensional convex body C containing the origin in its interior and a real $t > 1$, we seek to construct a polytope P with as few vertices as possible such that P is contained in C and C is contained in tP. I plan to present a construction which breaks some long-held records and is nearly optimal for a wide range of parameters d and t. The construction uses the maximum volume ellipsoid, the John decomposition of the identity and its recent sparsification by Batson, Spielman and Srivastava, Chebyshev polynomials, and some tensor algebra.

Saturday (April 20)
Morning session

9:45-10:25

Vladimir D. Tonchev (Michigan Technological University)

New invariants for incidence structures

Abstract. New isomorphism invariants for incidence structures based on a connection between trace codes and Galois geometry are discussed. Using these invariants, a new Hamada type characterization of the classical finite geometry designs is proved.

Saturday (April 20)
Morning session

10:50-11:10

Brittany Terese Fasy (Carnegie Mellon University)

with Sivaraman Balakrishnan, Fabrizio Lecci, Alessandro Rinaldo, Aarti Singh, and Larry Wasserman

Statistical inference for persistent homology

Abstract. Persistent homology is a method for probing topological properties of point clouds and functions. The method involves tracking the birth and death of topological features as one varies a tuning parameter. Features with short lifetimes are informally considered to be "topological noise." In this paper, we bring some statistical ideas to persistent homology. In particular, we derive confidence intervals that allow us to separate topological signal from topological noise.

11:15-11:35

David L. Millman (University of North Carolina at Chapel Hill)

Computing the discrete Voronoi diagram with only double precision

Abstract. The nearest neighbor transform of a binary image assigns to each pixel the index of the nearest black pixel – it is the discrete analog of the Voronoi diagram. Implementations that compute the transform use numerical calculations to perform geometric tests, so they may produce erroneous results if the calculations require more arithmetic precision than is available. Liotta, Preparata, and Tamassia, in 1999, suggested designing algorithms that not only minimize time and space resources, but also arithmetic precision.

A simple algorithm using double precision can compute the nearest neighbor transform: compare the squared distances of each pixel to all black pixels, but this is inefficient when many pixels are black. We develop and implement efficient algorithms, computing the nearest neighbor transform of an image in linear time with respect to the number of pixels, while still using only double precision.

Saturday (April 20)
Morning session

11:40-12:00

Alexander Magazinov (Steklov Mathematical Institute)

Fans of faces of parallelohedral tilings

Abstract.
A parallelohedron is a convex polytope P that admits a face-to-face tiling T_P of $\mathbb{R}^d$ by its translates.

The central conjecture concerning parallelohedra is the one by G. Voronoi (see [2]).

Conjecture 1. Every d-dimensional parallelohedron P is affinely equivalent to a Dirichlet-Voronoi domain for some d-dimensional lattice.

Conjecture 1 has not been proved or disproved so far in full generality. However, several significant partial results have been obtained. For many approaches the study of local structure of T_P is important.

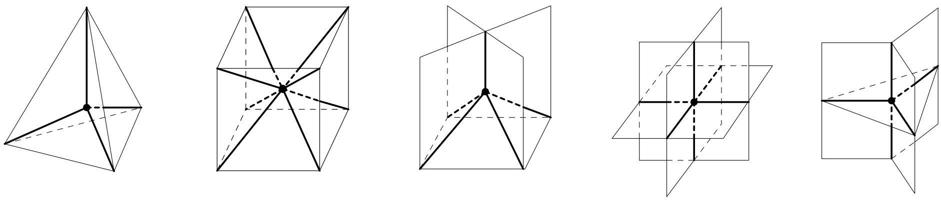
Definition 1. Denote by π the projection along $\mathrm{lin}F$ onto the complementary affine space $(\mathrm{lin}F)^{\mathrm{compl}}$. Then there exists a complete k-dimensional poyhedral fan $\mathrm{fan}(F)$ (*the fan of* F) that splits $(\mathrm{lin}F)^{\mathrm{compl}}$ into convex polyhedral cones with vertex $\pi(F)$, and a neighborhood $U = U(\pi(F))$ such that every face $F' \supset F$ corresponds to a cone $C \in \mathrm{fan}(F)$ satisfying

$$\pi(F') \cap U = C \cap U.$$

Remark 1. Speaking informally, $\mathrm{fan}(F)$ has the same combinatorial structure as the transversal section of T_P in a small neighborhood of F.

In 1929 B.N. Delaunay (see [1]) proved the key result.

Theorem 2. *Let P be a d-dimensional parallelohedron and F be a $(d-3)$-imensional face of T_P. Then fan(F) has one of the 5 combinatorial types shown in Figure 1. Moreover, each of these types is realized for some 3-imensional tiling.*



a) b) c) d) e)

Figure 1: 5 possible fans of $(d-3)$-faces

We give a combinatorial proof of Theorem 2. Also we prove a general result on fans of faces.

Definition 2. Let

$$\nu(F) = card\,\{P' \in \mathcal{T}(P) : F \subset P'\}.$$

$\nu(F)$ is called the *valence* of the face F.

Theorem 3. *Let P be a d-dimensional parallelohedron and F be a $(d-k)$-imensional face of T_P. Then*

$$\nu(F) \leq 2^k.$$

Theorem 3 immediately implies

Corollary 4. *Given $k \in \mathbb{N}$, there exists a set of complete k-dimensional polyhedral fans*

$$\{\mathcal{C}_1^k, \mathcal{C}_2^k, \ldots, \mathcal{C}_{N(k)}^k\}$$

such that for every d, every d-parallelohedron P and every $(d-k)$-face F of T_P the fan of F is isomorphic to some $\mathcal{C}_i^k$.

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