LOCAL COMPLEXITY OF FINITE SETS

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ABSTRACT.

By *local complexity* $\phi(X)$ of a finite subset $X \subset [0,1) = S^1$ of the unit circle we mean the cardinality of the set of distinct lengths of the intervals in the partition (of S^1) determined by X. We discuss various generalizations, old and new, of the 3-distance theorem claiming that

$$\phi\big(\{\langle k\alpha\rangle \mid 1 \leq k \leq n\}\big) \leq 3,$$

for all real α and $n \geq 1$. (Here $\langle x\rangle$ stands for the fractional part of x).

In other words, the 3-distance theorem claims that the finite pieces of orbits of rotations $R_\alpha \colon S^1 \to S^1$ have a uniform bound (namely, 3) on their local complexity. The result extends from rotations R_α to a larger class of transformations, in particular to all interval exchange transformations. (These transformations form a natural generalization of rotations and provide important examples in ergodic theory).

We review other examples when large sets with small local complexity appear. In particular [F. Dyson, 1991], for $\alpha = \sqrt[3]{2}$ and $\beta = \sqrt[3]{4}$,

$$\sup_{n\geq 1} \phi(\{\langle m\alpha + n\beta\rangle \Big| \ 1 \leq m, n \leq n\}) < \infty.$$

The above theorem extends for badly approximable pairs (α, β) in Schmidt's sense (but not to all pairs), see recent preprint by P. Bleher, R. Roeder, Y. Homma, L. Ji, and J. Shen (Arxiv 2011).

A number of open questions will be posed. (In particular, what is the right definition of the multidimensional local complexity?).