The distribution of reduced rationals in the unit interval

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Back story and main result

Sander, Meiss (2019):

Motivated by a numerical simulation in dynamical systems, sought to distinguish between computer generated approximations which represent rational numbers, and those which represent irrationals.

Basic question: If you choose a random interval of some fixed diameter, how large should you expect the smallest denominator of a fraction in this interval to be?

More precisely:

Fix $\delta \in (0,1)$, define $q_{min} : [0,1] \to \mathbb{N}$ by

$$q_{min}(x) = \min\{ q \in \mathbb{N} : \exists \tfrac{a}{q} \in \mathbb{Q} \cap (x - \delta/2, x + \delta/2) \}$$

Question: What is $E[q_{min}]$? (asymptotically as $\delta \to 0$)

Thm (Chen, H., 2021): As $\delta \to 0$,

$$E[q_{min}] = \frac{16}{\pi^2} \cdot \frac{1}{\delta^{1/2}} + O(\log^2 \delta).$$

Notation:

$f(x) = O(g(x))$, $x \in D \iff \exists C > 0$ s.t. $\forall x \in D$, $|f(x)| \leq C \cdot |g(x)|$

Previous related work

For $N \in \mathbb{N}$ and $1 \leq i \leq N$, define

$$r_{i,N} = \min\{r : \exists\, b/r \in \mathbb{Q} \cap (\tfrac{i-1}{N}, \tfrac{i}{N}]\},$$

and let $S(N) = \sum_{i=1}^{N} r_{i,N}$.

(can take $C_1 = 1/\pi^2$, $C_2 = 96$)

- Kruyswijk, Meijer (1977): $\exists\, C_1, C_2 > 0$ s.t. $\forall N \in \mathbb{N}$,

$$C_1 N^{3/2} \leq S(N) \leq C_2 N^{3/2}.$$

Conjecture: $S(N) \sim \frac{16}{\pi^2} \cdot N^{3/2}$ (consistent, heuristically, with our result)

- Stewart (2013): $1.35 \cdot N^{3/2} \leq S(N) \leq 2.04 \cdot N^{3/2}$.

- Balazard & Martin (March 2023):

$$S(N) = \frac{16}{\pi^2} \cdot N^{3/2} + O(N^{4/3} \log^2 N).$$

Proof uses our result, together with a detailed analysis which also relies on Weil's estimate for Kloosterman sums.

Background material

- Farey fractions of order $Q \in \mathbb{N}$:

$$\mathcal{F}_Q = \left\{ \frac{a}{q} : a, q \in \mathbb{Z},\ 0 \le a < q \le Q,\ (a,q)=1 \right\} \subseteq [0,1),$$

taken with the usual ordering in $[0,1)$.

$$\mathcal{F}_Q = \{ \gamma_1 < \gamma_2 < \cdots < \gamma_{\Phi(Q)} \}$$

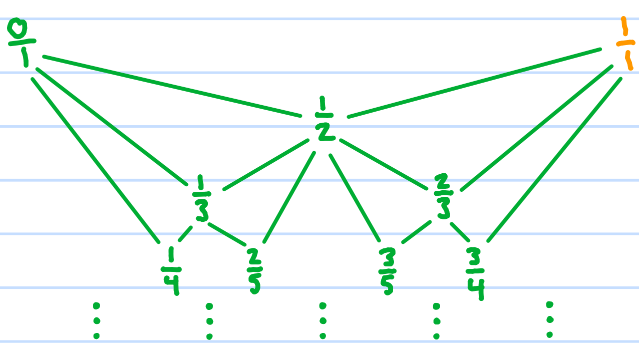
$$\# \mathcal{F}_Q = \sum_{q=1}^{Q} \varphi(q) = \Phi(Q) = \frac{3}{\pi^2} \cdot Q^2 + O(Q \log Q)$$

Euler phi function

Ex: $Q = 7$, $\Phi(Q) = 18$

$$\mathcal{F}_Q = \left\{ \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7} \right\}$$

Farey tree:



- If $\frac{b}{r} < \frac{a}{q}$ are consecutive in $\mathcal{F}_Q$ then $ar - bq = 1$.

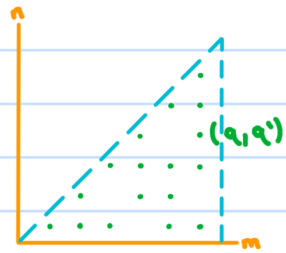
  This implies that $(q,r)=1$ and that $r \equiv a^{-1} \bmod q$.

- If $\frac{a'}{q'} < \frac{a}{q} < \frac{a''}{q''}$ are consecutive in $\mathcal{F}_q$ then $q = q' + q''$.

  So given $\frac{a}{q}$, we can compute $q' \equiv a^{-1} \bmod q$, $1 \le q' \le q$, and then $q'' = q - q'$.

- The map $\phi_Q : \mathcal{F}_Q \to \{(m,n) \in \mathbb{N}^2 : 1 \le n \le m \le Q,\ (m,n) = 1\}$ defined by $\phi_Q\left(\frac{a}{q}\right) = \begin{cases} (1,1) & \text{if } q = 1 \\ (q, q') & \text{if } q \ge 2 \end{cases}$ is a bijection.

$\frac{a}{q} \in \mathcal{F}_Q \longmapsto$



Ref: Hardy & Wright, <u>An Introduction to the Theory of Numbers</u>, Ch. 3.

- Distribution of $\mathcal{F}_Q$:
  - Uniform distribution: $\forall \alpha \in [0,1]$,

$$\#\{\gamma \in \mathcal{F}_Q : \gamma \in [0,\alpha]\} \sim \alpha\, \Phi(Q), \quad \text{as } Q \to \infty.$$

$\Rightarrow \forall$ Riemann integrable $f : [0,1] \to \mathbb{C}$,

$$\frac{1}{\Phi(Q)} \sum_{\gamma \in \mathcal{F}_Q} f(\gamma) \xrightarrow{Q \to \infty} \int_0^1 f(x)\,dx.$$

  - Discrepancy: $\forall \alpha \in [0,1]$,

$$\#\{\gamma \in \mathcal{F}_Q : \gamma \in [0,\alpha]\} = \alpha\, \Phi(Q) + O(Q \log Q).$$

Equivalently: $\forall \alpha \in [0,1]$,

$$\sum_{\gamma \in \mathcal{F}_Q} \left(\chi_{[0,\alpha]}(\gamma) - \alpha \right) \ll Q \log Q.$$

Notation: $f(x) \ll g(x)$ means $f(x) = O(g(x))$.

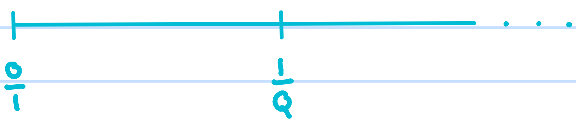
- Improvements: $\forall Q \in \mathbb{N}$, $\alpha \in [0,1]$, define

$$E_Q(\alpha) = \left| \sum_{\gamma \in \mathcal{F}_Q} \left(\chi_{[0,\alpha]}(\gamma) - \alpha \right) \right| .$$

- Niederreiter (1973): $\exists\, C_1, C_2 > 0$ s.t. $\forall Q \in \mathbb{N}$

$$C_1 Q \leq \sup_{\alpha \in [0,1]} E_Q(\alpha) \leq C_2 Q .$$

Lower bound:



$$E_Q\left(\tfrac{1}{Q}\right) = \left| 2 - \tfrac{1}{Q} \cdot \Phi(Q) \right| \gg Q .$$

- What about average values of $E_Q(\alpha)$?

$\forall Q \in \mathbb{N}$, define

$$\mathcal{E}_Q = \frac{1}{\Phi(Q)} \cdot \sum_{i=1}^{\Phi(Q)} E_Q(\gamma_i) .$$

Note:

$$\mathcal{E}_Q = \frac{1}{\Phi(Q)} \cdot \sum_{i=1}^{\Phi(Q)} \left| \sum_{\gamma \in \mathcal{F}_Q} \left(\chi_{[0,\gamma_i]}(\gamma) - \gamma_i \right) \right|$$

$$= \sum_{i=1}^{\Phi(Q)} \left| \frac{i}{\Phi(Q)} - \gamma_i \right|$$

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Neiderreiter: $\mathcal{E}_Q \ll Q$.

Best known: $\exists\, c > 0$ s.t. $\mathcal{E}_Q \ll Q \cdot \exp\left(\frac{-c \cdot (\log Q)^{0.6}}{(\log \log Q)^{0.2}} \right)$.

Conjecture 1: $\forall \varepsilon > 0$, $\mathcal{E}_Q \ll Q^{\frac{1}{2}+\varepsilon}$.

- Landau (1924): Conj. 1 $\Longleftrightarrow$ RH.

cf. Franel (1924)

Conjecture 2: $\exists\, \delta < 1$ s.t. $\mathcal{E}_Q \ll Q^{\delta}$.

Conjecture 3: $\forall \varepsilon > 0$, $E_Q(\frac{1}{3}) \ll Q^{\frac{1}{2}+\varepsilon}$. (easier?)

- Equiv. to RH on GRH for Dirichlet L-function with odd character mod 3.

<u>Back to main story</u>

$\delta \in (0,1)$, $q_{min}(x) = \min\{ q \in \mathbb{N} : \exists\, \frac{a}{q} \in \mathbb{Q} \cap (x - \delta/2, x + \delta/2) \}$

Thm (Chen, H., 2021): As $\delta \to 0$,

$$E[q_{min}] = \frac{16}{\pi^2} \cdot \frac{1}{\delta^{1/2}} + O(\log^2 \delta).$$

Sketch of proof:

Step 1: Compute the PMF of $q_{min} : [0,1] \to \mathbb{N}$.

PMF: $p : \mathbb{N} \to [0,1]$,

$$p(q) = \lambda\left(\{ x \in [0,1] : q_{min}(x) = q \}\right)$$

Step 2: Compute $E[q_{min}]$:

$$E[q_{min}] = \sum_{q \in \mathbb{N}} q \cdot p(q)$$

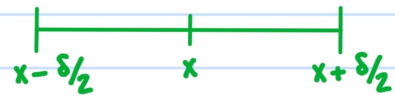
Step 1: Compute the PMF of $q_{\min} : [0,1] \to \mathbb{N}$.

$\delta \in (0,1)$, $\quad q_{\min}(x) = \min\{ q \in \mathbb{N} : \exists \frac{a}{q} \in \mathbb{Q} \cap (x - \delta/2, x + \delta/2) \}$

PMF: $p : \mathbb{N} \to [0,1]$,

$$p(q) = \lambda(\{x \in [0,1] : q_{\min}(x) = q\})$$

- Let $Q = \lfloor \delta^{-1} \rfloor + 1$. Then



$\forall x \in [0,1]$, $\quad q_{\min}(x) \in \{1, 2, \dots, Q\}$.

Therefore $p(q) = 0$ for $q > Q$.

- $\forall \gamma \in \mathcal{F}_Q$ define $I_\gamma \subseteq [0,1]$ by

$I_{0/1} = [0, \delta/2) \cup (1 - \delta/2, 1]$

$I_{a/q} = \{ x \in [0,1] : \frac{a}{q} \in (x - \delta/2, x + \delta/2), \ q_{\min}(x) = q \}$, if $q \geq 2$.

Not difficult to show that:

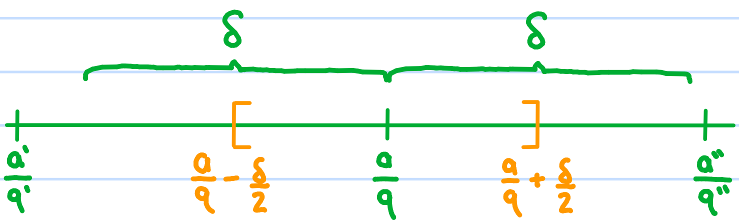
- For $q \geq 2$, $I_{a/q}$ is an interval.
- If $\gamma, \tilde{\gamma} \in \mathcal{F}_Q$, $\gamma \neq \tilde{\gamma}$, then

$$I_\gamma \cap I_{\tilde{\gamma}} = \emptyset.$$

- $\forall q \in \{1, \dots, Q\}$, $\quad p(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \lambda(I_{a/q})$.

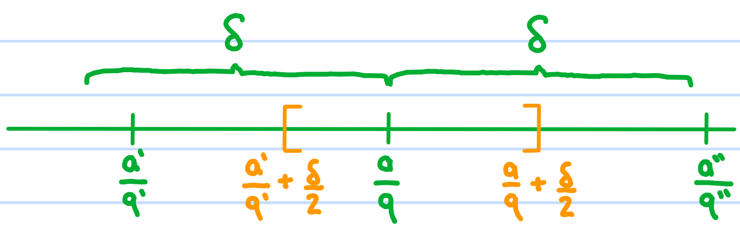
5 possibilities for $I_{a/q}$:

I)



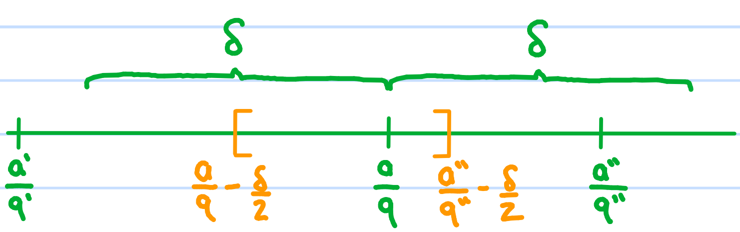
$$\lambda(I_{a/q}) = \delta$$

II a)



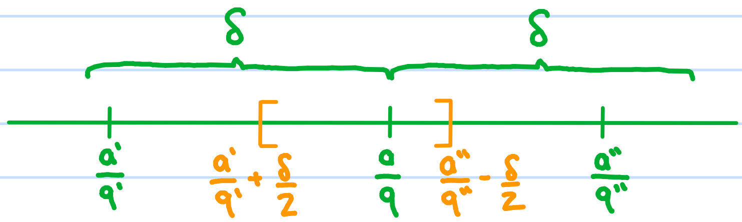
$$\lambda(I_{a/q}) = \left(\frac{a}{q}+\frac{\delta}{2}\right) - \left(\frac{a'}{q'}+\frac{\delta}{2}\right) = \frac{a}{q} - \frac{a'}{q'} = \frac{1}{qq'}$$

II b)



$$\lambda(I_{a/q}) = \frac{a''}{q''} - \frac{a}{q} = \frac{1}{qq''}$$

III)



$$\lambda\left(\overline{I}_{a/q}\right) = \left(\frac{a''}{q''} - \frac{\delta}{2}\right) - \left(\frac{a'}{q'} + \frac{\delta}{2}\right)$$

$$= \frac{1}{q'q''} - \delta = \frac{1}{q'(q-q')} - \delta$$

IV) $\frac{1}{q'q''} < \delta \Rightarrow \lambda\left(\overline{I}_{a/q}\right) = 0$

Thm: For $\delta \in (0,1)$, $p(1) = p(2) = \delta$,

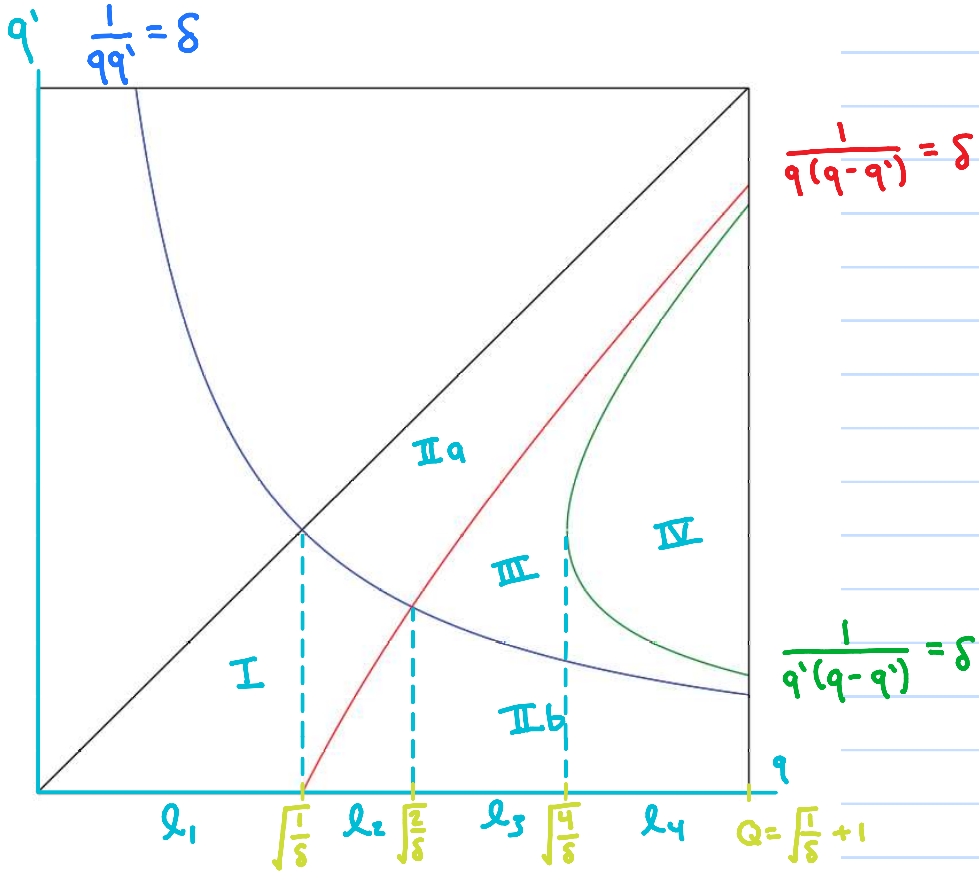
$$p(q) = \sum_{q'+q''=q}{}' \ \Pi\left(\frac{1}{qq'}, \frac{1}{qq''}; \delta\right), \quad q \geq 3,$$

where (Σ' means $(q', q) = 1$)

$$\Pi(\alpha, \beta; t) = \begin{cases} t & \text{if } t \leq \overline{\alpha}, \\ \overline{\alpha} & \text{if } \overline{\alpha} < t \leq \overline{\beta}, \\ \alpha + \beta - t & \text{if } \overline{\beta} < t \leq \alpha + \beta, \\ 0 & \text{if } t > \alpha + \beta, \end{cases}$$

$\overline{\alpha} = \min(\alpha, \beta)$ and $\overline{\beta} = \max(\alpha, \beta)$.

Step 2: Compute $E[q_{min}]$:



$$E[q_{min}] = \sum_{q \in \mathbb{N}} q \cdot p(q) = \sum_{i=1}^{4} \sum_{q \in \ell_i} q\, p(q).$$

Ex (easy):

$$\sum_{q \in \ell_1} q \overbrace{p(q)}^{=\delta \text{ if } q \in \ell_1} = \delta \sum_{q \le \sqrt{\frac{1}{\delta}}} q\, \varphi(q) = \delta \sum_{q \le \sqrt{\frac{1}{\delta}}} q^2 \sum_{d \mid q} \frac{\mu(d)}{d}$$

$$= \delta \sum_{d \le \sqrt{1/\delta}} d\, \mu(d) \sum_{r \le \frac{1}{d\sqrt{\delta}}} r^2$$

$$= \frac{1}{3\delta^{1/2}} \overbrace{\sum_{d \le \delta^{-1/2}} \frac{\mu(d)}{d^2}}^{\zeta^{-1}(2) + O(\delta^{-1/2})} + O\left(\sum_{d \le \delta^{-1/2}} \frac{|\mu(d)|}{d^2} \right)$$

$$= \frac{2}{\pi^2} \delta^{-1/2} + O(\log(1/\delta))$$

Other sums are progressively more difficult...

- $\displaystyle\sum_{q\in \ell_2} q\,p(q) = \frac{6\sqrt{2}}{\pi^2}\left(2\log 2 + \frac{7\sqrt{2}}{6} - \frac{8}{3}\right)\delta^{-1/2} + O(\log(1/\delta))$

- $\displaystyle\sum_{q\in \ell_3} q\,p(q) = \frac{6\sqrt{2}}{\pi^2}\left(4\sqrt{2}\log 2 - 2\log 2 + \frac{8}{3} - \frac{10\sqrt{2}}{3}\right)\delta^{-1/2} + O(\log(1/\delta))$

- $\displaystyle\sum_{q\in \ell_4} q\,p(q) = \frac{12}{\pi^2}\left(\frac{10}{3} - 4\cdot\log 2\right)\delta^{-1/2} + O(\log^2(\delta))$

Open problems

1) Find an asymptotic formula for the expected value of the smallest "denominator" of a rational point in a randomly chosen "ball" of fixed diameter in $\mathbb{R}^2$.

2) Find an asymptotic formula for the expected value of the smallest denominator of an element of $\mathbb{Q}(i)$ in a randomly chosen ball in $\mathbb{C}$.