The derivative NLS system and its solution with the help of the Marchenko method

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Outline

1. inverse scattering problems
2. Marchenko method to solve inverse scattering problems
3. Inverse Scattering Transform method to solve integrable systems
4. Marchenko method to solve integrable systems
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7. reduction in the DNLS system, the Kaup–Newell equation (DNLS I)
8. Marchenko method to solve the semidiscrete DNLS system
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Inverse scattering problems

- $-\psi''(k,x) + V(x)\,\psi(k,x) = k^2\,\psi(k,x), \qquad x \in \mathbb{R}^+$
- $\mathbb{R}^+ := (0, +\infty)$, half line, $x > 0$; prime: x-derivative
- k^2 : spectral parameter
- potential $V(x)$: decaying as $x \to +\infty$
- physical solution ψ: $\psi(k,x) = e^{-ikx} + S(k)\,e^{ikx} + o(1), \qquad x \to +\infty$
- scattering function $S(k)$
- inverse scattering problem: given $S(k)$ for all $k > 0$, determine $V(x)$ for all $x > 0$
- other inverse scattering problems for differential/difference equations/systems
- inverse scattering problems: given $S(k,t)$ for all $k > 0$, determine $V(x,t)$ for all $x > 0$

The Marchenko method

- Vladimir A. Marchenko, from Ukraine, 100th anniversary
- one-to-one correspondence between $S(k,t)$ and $V(x,t)$
- use a Fourier transform on $S(k,t)$ and obtain $F(y,t)$
- use $F(y,t)$ as input to the linear Marchenko integral equation
- $K(x,y,t) + F(x+y,t) + \int_x^\infty dz\, K(x,z,t)\, F(z+y,t) = 0, \qquad y > x, \quad t \in \mathbb{R}$
- from the solution $K(x,y,t)$, use $K(x,x^+,t)$ to recover $V(x,t)$ at each fixed t
- Marchenko method exists for some differential/difference equations/systems
- no Marchenko method for many other differential/difference equations/systems

Application to integrable nonlinear systems

- Inverse Scattering Transform method to solve IVPs for integrable nonlinear systems
- associate the nonlinear system with a linear system
- solution $u(x,t)$ to the nonlinear system appears as a coefficient in the linear system
- goal is to obtain the solution $u(x,t)$ when $u(x,0)$ is given
- use the Inverse Scattering Transform method to solve $u(x,0) \mapsto u(x,t)$

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{direct scattering for linear system at } t=0} & S(k,0) \\ \text{solution to nonlinear system} \Big\downarrow & & \Big\downarrow \text{time evolution of scattering data} \\ u(x,t) & \xleftarrow[\text{inverse scattering for linear system at time } t]{} & S(k,t) \end{array}$$

The Marchenko method to solve integrable systems

- extend the Inverse Scattering Transform method

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{direct scattering at } t=0"] \arrow[d, "\text{solution}"'] & S(k,0) \arrow[r, "\text{Fourier transform at } t=0"] \arrow[d, "\text{time evolution}"] & F(y,0) \arrow[d, "\text{time evolution}"] \\
u(x,t) & S(k,t) \arrow[l, "\text{inverse scattering at time } t"] & F(y,t) \arrow[l, "\text{inverse Fourier transform at time } t"]
\end{tikzcd}$$

- recover the potential from the solution to the Marchenko equation

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{Marchenko kernel at } t=0"] \arrow[d, "\text{solution}"'] & F(y,0) \arrow[r, "\text{time evolution}"] & F(y,t) \arrow[d, "\text{input to Marchenko equation}"] \\
u(x,t) & K(x,x^+,t) \arrow[l, "\text{recover the potential}"] & K(x,y,t) \arrow[l, "\text{recover the potential}"]
\end{tikzcd}$$

The general DNLS system

- linear system for general DNLS (three complex parameters a, b, κ)

$$\frac{d}{dx}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}-i\lambda+(ib/2)\,qr & \kappa\sqrt{\lambda}\,q\\ (1/\kappa)\sqrt{\lambda}\,r & i\lambda+(ia/2)\,qr\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix},\qquad x,t\in\mathbb{R}.$$

- DNLS system [two parameters $(a-b)$ and κ]

$$\begin{cases} i\,q_t+q_{xx}+i\kappa(a-b-2)qq_xr+i\kappa(a-b-1)q^2r_x+\dfrac{\kappa^2(a-b)(a-b-1)}{4}q^3r^2=0,\\ i\,r_t-r_{xx}+i\kappa(a-b-2)qrr_x+i\kappa(a-b-1)q_xr^2-\dfrac{\kappa^2(a-b)(a-b-1)}{4}\,q^2r^3=0.\end{cases}$$

$$(a,b,\kappa)=\begin{cases}(0,0,1), & \text{(DNLS I, Kaup–Newell system)},\\ (1,0,1), & \text{(DNLS II, Chen–Lee–Liu system)},\\ (1,-1,1), & \text{(DNLS III, Ivanov–Gerdjikov system)}.\end{cases}$$

- Marchenko method with (a,b,κ), or Marchenko with $(0,0,1)$ and transform to (a,b,κ)

$$q(x,t)\mapsto\frac{1}{\kappa}\,q(x,t)\,E(x,t)^{b-a},\qquad r(x,t)\mapsto\kappa\,r(x,t)\,E(x,t)^{a-b},$$

$$E(x,t):=\exp\left(\frac{i}{2}\int_{-\infty}^{x}dz\,q(z,t)\,r(z,t)\right).$$

The DNLS system (DNLS I, Kaup–Newell)

- linear system associated with the DNLS system

$$\frac{d}{dx}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}-i\lambda & \sqrt{\lambda}\,q\\ \sqrt{\lambda}\,r & i\lambda\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix},\qquad x,t\in\mathbb{R}.$$

- DNLS system

$$\begin{cases} iq_t+q_{xx}-i(qrq)_x=0,\\ ir_t-r_{xx}-i(rqr)_x=0,\end{cases}\qquad x,t\in\mathbb{R}.$$

- AKNS system associated with the NLS system

$$\frac{d}{dx}\begin{bmatrix}\xi\\ \eta\end{bmatrix}=\begin{bmatrix}-i\lambda & u\\ v & i\lambda\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix},\qquad x,t\in\mathbb{R}.$$

- NLS system

$$\begin{cases} iu_t+u_{xx}-2u^2v=0,\\ iv_t-v_{xx}+2uv^2=0,\end{cases}\qquad x,t\in\mathbb{R}.$$

The Marchenko method for the DNLS system

- linear system of Marchenko integral equations

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{K}_1(x,y,t) & K_1(x,y,t) \\ \bar{K}_2(x,y,t) & K_2(x,y,t) \end{bmatrix} + \begin{bmatrix} 0 & \bar{\Omega}(x+y,t) \\ \Omega(x+y,t) & 0 \end{bmatrix}$$

$$+ \int_x^\infty dz \begin{bmatrix} -iK_1(x,z,t)\,\Omega'(z+y,t) & \bar{K}_1(x,z,t)\,\bar{\Omega}(z+y,t) \\ K_2(x,z,t)\,\Omega(z+y,t) & i\bar{K}_2(x,z,t)\,\bar{\Omega}'(z+y,t) \end{bmatrix}, \qquad x<y.$$

- reflection coefficients $R(\sqrt{\lambda},0)$ and $\bar{R}(\sqrt{\lambda},0)$
- bound-state data via the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$

$$\begin{cases} \Omega(y,t) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} d\lambda\, \dfrac{R(\sqrt{\lambda},0)}{\sqrt{\lambda}}\, e^{4i\lambda^2 t}\, e^{i\lambda y} + C\, e^{4iA^2 t}\, e^{iAy}\, B, \\ \bar{\Omega}(y,t) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} d\lambda\, \dfrac{\bar{R}(\sqrt{\lambda},0)}{\sqrt{\lambda}}\, e^{-4i\lambda^2 t}\, e^{-i\lambda y} + \bar{C}\, e^{-4i\bar{A}^2 t}\, e^{-i\bar{A}y}\, \bar{B}. \end{cases}$$

The Marchenko method for the NLS system

- linear system of Marchenko integral equations

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{K}_1(x,y,t) & K_1(x,y,t) \\ \bar{K}_2(x,y,t) & K_2(x,y,t) \end{bmatrix} + \begin{bmatrix} 0 & \bar{\Omega}(x+y,t) \\ \Omega(x+y,t) & 0 \end{bmatrix}$$

$$+ \int_x^\infty dz \begin{bmatrix} K_1(x,z,t)\,\Omega(z+y,t) & \bar{K}_1(x,z,t)\,\bar{\Omega}(z+y,t) \\ K_2(x,z,t)\,\Omega(z+y,t) & \bar{K}_2(x,z,t)\,\bar{\Omega}(z+y,t) \end{bmatrix}, \qquad x < y.$$

- reflection coefficients $R(\lambda,0)$ and $\bar{R}(\lambda,0)$
- bound-state data via the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$
- use of (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ allows any number of bound states with any multiplicities

$$\begin{cases} \Omega(y,t) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} d\lambda\, R(\lambda,0)\, e^{4i\lambda^2 t}\, e^{i\lambda y} + C\, e^{4iA^2 t}\, e^{iAy}\, B, \\ \bar{\Omega}(y,t) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} d\lambda\, \bar{R}(\lambda,0)\, e^{-4i\lambda^2 t}\, e^{-i\lambda y} + \bar{C}\, e^{-4i\bar{A}^2 t}\, e^{-i\bar{A}y}\, \bar{B}. \end{cases}$$

Recovery via the Marchenko method for the DNLS system

- recovery of potentials $q(x,t)$ and $r(x,t)$ and the key quantity $E(x,t)$

$$\begin{cases} q(x,t) = -2K_1(x,x,t)\exp\left(-4\int_x^\infty dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right), \\ r(x,t) = -2\bar{K}_2(x,x,t)\exp\left(4\int_x^\infty dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right), \\ E(x,t) = \exp\left(2\int_{-\infty}^x dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right). \end{cases}$$

- reflectionless case: separable-kernel Marchenko system and hence explicit solutions
- closed-form, compact formulas for explicit solutions involving matrix exponentials
- “unpacking” matrix exponentials yields explicit solutions in terms of elementary functions
- animation of explicit solutions via Mathematica

The reductions $r = \pm q^*$ in the Marchenko method for DNLS

- reduced Marchenko equation in one dependent variable $K_1(x, y, t)$

$$K_1(x, y, t) \pm \Omega(x + y, t) \pm i \int_x^\infty dz\, K_1(x, z, t)\, \Omega'(z + s, t)\, \Omega(s + y, t)^* = 0, \qquad y > x.$$

- input $\Omega(y, t)$ to the Marchenko equation

$$\Omega(y, t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \frac{R(\sqrt{\lambda}, 0)}{\sqrt{\lambda}}\, e^{4i\lambda^2 t}\, e^{i\lambda y} + C\, e^{4iA^2 t}\, e^{iAy}\, B.$$

- recovery of the potential $q(x, t)$

$$q(x, t) = -2K_1(x, x, t)\, \exp\left(\mp 4i \int_x^\infty dz\, |K_1(z, z, t)|^2 \right).$$

- reflectionless case: input to the Marchenko equation

$$\Omega(y, t) = C\, e^{4iA^2 t}\, e^{iAy}\, B.$$

- kernel $\Omega(z + y, t)$ separable in z and y in the reflectionless case

$$\Omega(z + y, t) = C\, e^{4iA^2 t}\, e^{iAz}\, e^{iAy}\, B.$$

- closed-form, compact formulas for explicit solutions, animations via Mathematica

Reflectionless case: Explicit solutions for the DNLS system

- use (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as input to the Marchenko system
- obtain the Marchenko solution $K_1(x, y, t)$, $K_2(x, y, t)$, $\bar{K}_1(x, y, t)$, $\bar{K}_2(x, y, t)$.
- evaluate $K_1(x, x, t)$, $K_2(x, x, t)$, $\bar{K}_1(x, x, t)$, $\bar{K}_2(x, x, t)$.
- obtain $q(x, t)$ and $r(x, t)$ via

$$\begin{cases} q(x,t) = -2K_1(x,x,t) \exp\left(-4\int_x^\infty dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right), \\ r(x,t) = -2\bar{K}_2(x,x,t) \exp\left(4\int_x^\infty dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right). \end{cases}$$

- animation of $|q(x, t)|$ and $|r(x, t)|$ using Mathematica
- Mathematica notebook using (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as input

Reflectionless case: Solution steps for the DNLS system

- construct M and $\bar M$ by solving

$$AM - M\bar A = iB\bar C, \qquad \bar M A - \bar A\bar M = i\bar B C.$$

- construct $\Gamma(x,t)$ and $\bar\Gamma(x,t)$ via

$$\begin{cases} \Gamma(x,t) := I - e^{iAx+4iA^2t} M\bar A e^{-2i\bar Ax-4i\bar A^2t}\bar M e^{iAx}, \\ \bar\Gamma(x,t) := I - e^{-i\bar Ax-4i\bar A^2t}\bar M A e^{2iAx+4iA^2t} M e^{-i\bar Ax}. \end{cases}$$

- construct $K_1(x,y,t)$, $K_2(x,y,t)$, $\bar K_1(x,y,t)$, and $\bar K_2(x,y,t)$ via

$$\begin{cases} K_1(x,y,t) = -\bar C\, e^{-i\bar Ax}\,\bar\Gamma(x,t)^{-1}\, e^{-i\bar Ay-4i\bar A^2t}\,\bar B, \\ K_2(x,y,t) = C\, e^{iAx}\,\Gamma(x,t)^{-1}\, e^{iAx+4iA^2t} M\bar A e^{-i\bar A(x+y)-4i\bar A^2t}\,\bar B, \\ \bar K_1(x,y,t) = \bar C\, e^{-i\bar Ax}\,\bar\Gamma(x,t)^{-1}\, e^{-i\bar Ax-4i\bar A^2t}\bar M A\, e^{iA(x+y)+4iA^2t}\, B, \\ \bar K_2(x,y,t) = -C\, e^{iAx}\,\Gamma(x,t)^{-1}\, e^{iAy+4iA^2t}\, B. \end{cases}$$

- obtain $q(x,t)$ and $r(x,t)$ explicitly in terms of (A,B,C) and $(\bar A,\bar B,\bar C)$

Recovery via the Marchenko method for the NLS system

- recovery of potentials $q(x,t)$ and $r(x,t)$ and the key quantity $E(x,t)$

$$\begin{cases} u(x,t) = -2K_1(x,x,t), \\ v(x,t) = -2\bar{K}_2(x,x,t). \end{cases}$$

- reflectionless case: separable-kernel Marchenko system and hence explicit solutions
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- animation of explicit solutions via Mathematica

The discrete DNLS system

- discrete linear system associated with the semidiscrete derivative NLS system

$$\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}=\begin{bmatrix} z & (z-1/z)\,q_n\\ z\,r_n & 1/z+(z-1/z)\,q_n\,r_n\end{bmatrix}\begin{bmatrix}\alpha_{n+1}\\ \beta_{n+1}\end{bmatrix},\qquad n\in\mathbb{Z},\quad t\in\mathbb{R}.$$

- semidiscrete DNLS system

$$\begin{cases} i\dot{q}_n+\dfrac{q_{n+1}}{1-q_{n+1}r_{n+1}}-\dfrac{q_n}{1-q_nr_n}-\dfrac{q_n}{1+q_nr_{n+1}}+\dfrac{q_{n-1}}{1+q_{n-1}r_n}=0,\\[2mm] i\dot{r}_n-\dfrac{r_{n+1}}{1+q_nr_{n+1}}+\dfrac{r_n}{1+q_{n-1}r_n}+\dfrac{r_n}{1-q_nr_n}-\dfrac{r_{n-1}}{1-q_{n-1}r_{n-1}}=0.\end{cases}$$

- discrete linear system associated with the semidiscrete NLS system

$$\begin{bmatrix}\xi_n\\ \eta_n\end{bmatrix}=\begin{bmatrix} z & z\,u_n\\ (1/z)\,v_n & 1/z\end{bmatrix}\begin{bmatrix}\xi_{n+1}\\ \eta_{n+1}\end{bmatrix},\qquad n\in\mathbb{Z},\quad t\in\mathbb{R}.$$

- semidiscrete NLS system

$$\begin{cases} i\dot{u}_n+u_{n+1}-2u_n+u_{n-1}-u_nu_{n+1}v_n-u_{n-1}u_nv_n=0,\\ i\dot{v}_n-v_{n+1}+2v_n-v_{n-1}+u_nv_nv_{n+1}+u_nv_{n-1}v_n=0.\end{cases}$$

The Marchenko method for the semidiscrete DNLS system

- linear system of Marchenko integral equations

$$\begin{bmatrix} \bar{K}_{nm}^{(1)} & K_{nm}^{(1)} \\ \bar{K}_{nm}^{(2)} & K_{nm}^{(2)} \end{bmatrix} + \begin{bmatrix} 0 & \bar{\Omega}_{n+m} \\ \Omega_{n+m} & 0 \end{bmatrix}$$
$$+ \sum_{l=n+1}^{\infty} \begin{bmatrix} \bar{K}_{nl}^{(1)} & K_{nl}^{(1)} \\ \bar{K}_{nl}^{(2)} & K_{nl}^{(2)} \end{bmatrix} \begin{bmatrix} 0 & \bar{\Omega}_{l+m} \\ \Omega_{l+m} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad m > n.$$

- reflection coefficients $R(z,0)$ and $\bar{R}(z,0)$
- bound-state data via the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$

$$\begin{cases} \Omega_k := \dfrac{1}{2\pi i} \displaystyle\oint dz\, R\, e^{-it(z-z^{-1})^2}\, z^{k-1} + C\, e^{-it(A-A^{-1})^2} A^{k-1} B, & k \text{ even}, \\ \bar{\Omega}_k := \dfrac{1}{2\pi i} \displaystyle\oint dz\, \bar{R}\, e^{it(z-z^{-1})^2}\, z^{-k-1} + \bar{C}\, e^{it[\bar{A}-(\bar{A})^{-1}]^2}\, (\bar{A})^{-k-1} \bar{B}, & k \text{ even}, \\ \Omega_k := 0, \quad \bar{\Omega}_k := 0, & k \text{ odd}. \end{cases}$$

The Marchenko method for the semidiscrete NLS system

- linear system of Marchenko integral equations

$$\begin{bmatrix} \bar{K}_{nm}^{(1)} & K_{nm}^{(1)} \\ \bar{K}_{nm}^{(2)} & K_{nm}^{(2)} \end{bmatrix} + \begin{bmatrix} 0 & \bar{\Omega}_{n+m} \\ \Omega_{n+m} & 0 \end{bmatrix} + \sum_{l=n+1}^{\infty} \begin{bmatrix} \bar{K}_{nl}^{(1)} & K_{nl}^{(1)} \\ \bar{K}_{nl}^{(2)} & K_{nl}^{(2)} \end{bmatrix} \begin{bmatrix} 0 & \bar{\Omega}_{l+m} \\ \Omega_{l+m} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad m > n.$$

- reflection coefficients $R(z,0)$ and $\bar{R}(z,0)$
- bound-state data via the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$

$$\begin{cases} \Omega_k := \dfrac{1}{2\pi i} \displaystyle\oint dz\, R\, e^{-it(z-z^{-1})^2}\, z^{k-1} + C\, e^{-it(A-A^{-1})^2} A^{k-1} B, & k \text{ even}, \\ \bar{\Omega}_k := \dfrac{1}{2\pi i} \displaystyle\oint dz\, \bar{R}\, e^{it(z-z^{-1})^2}\, z^{-k-1} + \bar{C}\, e^{it[\bar{A}-(\bar{A})^{-1}]^2} (\bar{A})^{-k-1} \bar{B}, & k \text{ even}, \\ \Omega_k := 0, \quad \bar{\Omega}_k := 0, \qquad k \text{ odd}. \end{cases}$$

Recovery of potentials in the semidiscrete DNLS system

- recovery of potentials q_n and r_n

$$q_n = \frac{\displaystyle\sum_{l=n}^{\infty} K_{nl}^{(1)} \sum_{k=n}^{\infty} K_{nk}^{(2)}}{\displaystyle\sum_{l=n}^{\infty} \bar{K}_{nl}^{(1)} \sum_{k=n}^{\infty} K_{nk}^{(2)} - \sum_{l=n}^{\infty} K_{nl}^{(1)} \sum_{k=n}^{\infty} \bar{K}_{nk}^{(2)}}, \qquad r_n = \frac{\displaystyle\sum_{l=n-1}^{\infty} \bar{K}_{(n-1)l}^{(2)}}{\displaystyle\sum_{l=n-1}^{\infty} K_{(n-1)l}^{(2)}} - \frac{\displaystyle\sum_{l=n}^{\infty} \bar{K}_{nl}^{(2)}}{\displaystyle\sum_{l=n}^{\infty} K_{nl}^{(2)}}.$$

- reflectionless case: separable-kernel Marchenko system and hence explicit solutions
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Recovery of potentials in the semidiscrete NLS system

- recovery of potentials q_n and r_n

$$u_n = K^{(1)}_{n(n+2)}, \qquad v_n = \bar{K}^{(2)}_{n(n+2)}.$$

- reflectionless case: separable-kernel Marchenko system and hence explicit solutions
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Relevant references

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