Lax Representation For Camassa-Holm Hierarchy

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Abstract

We study the Camassa-Holm (CH) hierarchy and its Lax representation. The CH hierarchy is an integrable hierarchy of nonlinear evolution equations. It includes the well-known Camassa-Holm equation, a model for the propagation of waves in shallow water. We start from the CH spectral problem and find Lenard's operators. Then, we construct the recursion operator, which gives rise to Lenard's sequences and the CH vector fields. Finally, through finding the solution to a key matrix equation, we obtain the Lax representation for the hierarchy.

Spectral Problem

The Camassa-Holm spectral problem is given by:

$$\psi_{xx} = \frac{1}{4}\psi - \frac{1}{2}m\lambda\psi$$

where λ is an eigenvalue, ψ is the corresponding eigenfunction, m is a potential function, x is the spatial variable, and subscripts denote partial derivatives.

Lenard's Operators

To find Lenard's operators we set $\nabla\lambda = \psi^2$, where $\nabla\lambda$ represents the gradient of the eigenvalue λ with respect to the potential function m, and search for the operators K and J which satisfy: $K \cdot \nabla\lambda = \lambda J \cdot \nabla\lambda$.

$$\Rightarrow (\nabla\lambda)_x = 2\psi\psi_x$$

$$\Rightarrow (\nabla\lambda)_{xx} = 2(\psi_x^2 + \psi^2(\tfrac{1}{4} - \tfrac{1}{2}m\lambda))$$

$$\Rightarrow (\nabla\lambda)_{xxx} = (1 - 2m\lambda)(\nabla\lambda)_x - m_x\lambda(\nabla\lambda)$$

$$\Rightarrow (-\partial^3 + \partial)\cdot\nabla\lambda = \lambda(\partial m + m\partial)\cdot\nabla\lambda$$

So, Lenard's operators are $K = -\partial^3 + \partial$ and $J = \partial m + m\partial$, where ∂ represents a partial derivative with respect to x.

Recursion Operator

The recursion operator is given by

$$\mathcal{L} = J^{-1}K = \tfrac{1}{2}m^{-\frac{1}{2}}\partial^{-1}m^{-\frac{1}{2}}(\partial - \partial^3)$$

where J^{-1} is the inverse of J. We also find

$$\mathcal{L}^{-1} = -e^{-x}\partial^{-1}e^{2x}\partial^{-1}e^{-x}\partial^{-1}(\partial m + m\partial)$$

where ∂^{-1} is the integral operator.

Lenard's Sequences

First, we take an element G_0 from the kernel of J.

$$Ker\ J = \{G \mid J \cdot G = 0\}$$

$$\Rightarrow J \cdot G_0 = (2m^{\frac{1}{2}}(m^{\frac{1}{2}}G_0)_x) = 0$$

$$\Rightarrow G_0 = m^{-\frac{1}{2}}$$

We also take an element G_{-1} from the kernel of K.

$$Ker\ K = \{G \mid K \cdot G = 0\}$$

$$\Rightarrow K \cdot G_{-1} = (G_{-1})_x - (G_{-1})_{xxx} = 0$$

$$\Rightarrow G_{-1} = a + be^x + ce^{-x}$$

where a, b, c are constants. We select $a = -1, b = 0, c = 0$, since those values lead to the CH equation.

$$\Rightarrow G_{-1} = -1$$

Next, we define Lenard's sequences as:

$$G_j = \begin{cases} \mathcal{L}^j \cdot G_0, & j \geq 0 \\ \mathcal{L}^{j+1} \cdot G_{-1}, & j < 0 \end{cases}$$

where j is an integer.

Camassa-Holm Hierarchy

The Camassa-Holm hierarchy is defined as:

$$m_{t_k} = J \cdot G_k, \quad k = 0, 1, 2, \ldots$$

where each k gives a different nonlinear evolution equation. For example:

$$m_{t_1} = (m^{-\frac{1}{2}})_x - (m^{-\frac{1}{2}})_{xxx}$$

is a Dym type equation. Also, by setting $m = u - u_{xx}$, we get

$$m_{t_{-2}} = -2mu_x - m_x u$$

which is the Camassa-Holm equation.

Lax Representation

To find the Lax form, we first solve the following matrix equation for $V(G)$:

$$V_x - [U, V] = U_*(K \cdot G - \lambda J \cdot G)$$

where

$$U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} - \frac{1}{2}m\lambda & 0 \end{pmatrix},$$

U_* is its directional derivative, G is an arbitrary function, and [ ] represents the commutator. This matrix equation has the solution:

$$V(G) = \lambda \begin{pmatrix} -\frac{1}{2}G_x & -G \\ \frac{1}{2}G_{xx} - \frac{1}{4}G + \frac{1}{2}m\lambda G & \frac{1}{2}G_x \end{pmatrix}.$$

Finally, the Lax representation for the CH hierarchy is given by:

$$U_{t_k} - V_{k,x} + [U, V_k] = 0, \ \ k \in \mathbb{Z}$$

where

$$V_k = \sum V(G_j)\lambda^{k-j-1}, \qquad \sum = \begin{cases} \sum_{j=0}^{k-1}, & k > 0 \\ 0, & k = 0 \\ -\sum_{j=k}^{-1}, & k < 0 \end{cases}$$

Qiao, Z. (2003). The Camassa-Holm Hierarchy, N -Dimensional Integrable Systems, and Algebro-Geometric Solution on a Symplectic Submanifold. *Communications in Mathematical Physics*, 239(1-2), 309-341.