Recursive Polynomials

Forms & patterns of the Golden-type

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What to expect

1. **Important definitions**
2. **Fibonacci polynomials**
3. **Golden polynomials**
4. **Our work - generalizing Golden polynomials**
   - Forms
   - Patterns
5. **Further study**

Defining recursion

- a starting step
- how each step builds on each other
- when to stop stepping

Figure: The larger Russian dolls have room inside them for the smaller dolls; the larger recursive function defines smaller recursive functions inside of itself.

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Recursion example - Fibonacci Numbers

Definition

Given $F_0 = 1$ and $F_1 = 1$, we define **Fibonacci numbers** as:

$$\forall\ n \geq 2:\ \ F_n = F_{n-1} + F_{n-2}$$

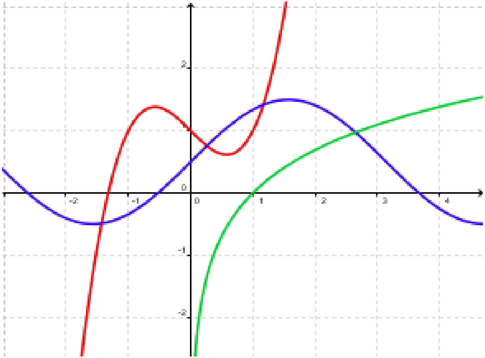
For example:

$$\begin{aligned}
F_2 &= F_2 + F_1 = 1 + 1 = 2 \\
F_3 &= F_3 + F_2 = 2 + 1 = 3 \\
F_4 &= F_4 + F_3 = 3 + 2 = 5 \\
F_5 &= F_5 + F_4 = 5 + 3 = 8
\end{aligned}$$

Defining polynomials

- poly means many
- nomial means term

Polynomials are some function of x with many terms!



Defining recursive polynomials

- at least 1 starting term
- some function of x, usually defined by $\gamma(x)$, that builds on terms
- an $n \in \mathbb{N}$ that defines when the function stops

Figure: The arrangement of seeds in a sunflower can be described recursively

Fibonacci polynomials

Definition

Given initial conditions $F_0 = 1$ and $F_1 = x$ and $\gamma(x) = x$ as a function of x, define a **Fibonacci polynomial** as:

$$\forall\, n \geq 2 : \; F_n(x) = x * F_{n-1}(x) + F_{n-2}(x)$$

For Example:

$$F_2 = x^2 + 1$$
$$F_3 = x^3 + 2x$$
$$F_4 = x^4 + 3x^2 + 3x$$
$$F_5 = x^5 + 4x^3 + 3x$$

Formula for Fibonacci polynomials

Below are formulas that generate even indices of Fibonacci polynomials:

$$F_{2n}(x) = \sum_{k=0}^{k=n} \binom{2n-k}{k} x^{2n-2k}$$

Binomial expansion

$$F_{2n} = \sum_{k=0}^{k=n} \frac{1}{k!} * \frac{d^k}{dx^k}(x^{2n-k})$$

Taylor expansion

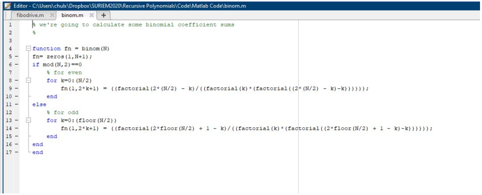


Figure: MATLAB code verified results

Roots for Fibonacci polynomials

The following formula represents the exact expression of roots of $F_n(x)$:

$$\text{When } F_n(x) = 0:$$

$$x = 2i\cos\frac{k\pi}{n}$$

$$\text{for } k = 1, 2, 3, \ \ldots, \ n-1$$

Hogatt discovered this unique result for Fibonacci polynomials.

Golden polynomials

Definition

Given initial conditions $G_0 = -1$ and $G_1 = x - 1$ and $\gamma(x) = x$ as a function of x, define **Golden polynomials** as:

$$\forall\ n \geq 2: \ G_n(x) = x * G_{n-1}(x) + G_{n-2}(x)$$

For Example:

$$\begin{aligned} G_2 &= x^2 - x - 1 \\ G_3 &= x^3 - x^2 - 1 \\ G_4 &= x^4 - x^3 + x^2 - 2x - 1 \\ G_5 &= x^5 - x^4 + 2x^3 - 3x^2 - x - 1 \end{aligned}$$

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Roots for Golden polynomials

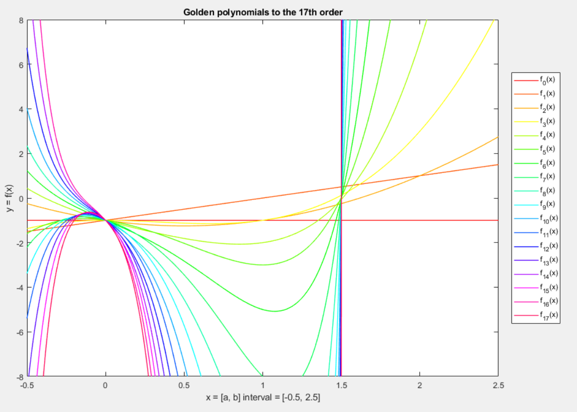

Figure: Moore found the limit of the maximum roots of $G_n(x)$ is $\frac{3}{2}$

Generalizing Golden polynomials

Alternate Definition

Given initial conditions $G_0 = -1$ and $G_1 = x - 1$ and $\gamma(x) = x$ as a function of x, redefine **Golden polynomials** as:

$$\begin{aligned}\forall\, n \geq 2 : G_n(x) &= x^1 * G_{n-1}(x) + x^0 * G_{n-2}(x)\\ &= x * G_{n-1}(x) + 1 * G_{n-2}(x)\\ &= x * G_{n-1}(x) + G_{n-2}(x)\end{aligned}$$

Then we can describe some function x^1 multiplied by $G_{n-1}(x)$ term, and some function x^0 multiplied by $G_{n-2}(x)$ term.

Generalized $G_n(x)$ in terms of k and l

Definition

With initial conditions $G_0(x) = -1$ and $G_1(x) = x - 1$, generalize the **Golden polynomials** as:

$$\forall\ n \geq 2 : G_n(x) = x^k * G_{n-1}(x) + x^l * G_{n-2}(x)$$

We have already shown the example for $k = 1$ and $l = 0$. When $k = l = 1$,

$$G_n(x) = x * (G_{n-1}(x) + G_{n-2}(x))$$

$G_n(x)$ for $k = l = 1$

$$
\begin{aligned}
G_0 &= -1 \\
G_1 &= x - 1 \\
G_2 &= x^2 - 2x \\
G_3 &= x^3 - x^2 - x \\
G_4 &= x^4 - 3x^2 \\
G_5 &= x^5 + x^4 - 4x^3 - 2x^2 \\
G_6 &= x^6 + 2x^5 - 4x^4 - 4x^3 \\
G_7 &= x^7 + 3x^6 - 3x^5 - 8x^4 - x^3 \\
G_8 &= x^8 + 4x^7 - x^6 - 12x^5 - 5x^4 \\
G_9 &= x^9 + 5x^8 + 2x^7 - 15x^6 - 13x^5 - x^4
\end{aligned}
$$

Pascal triangles in $F_n(x)$

Coefficients of Fibonacci polynomial form a Pascal 2-triangle.

Table 3
The Pascal 2-triangle

1.							1						
2.							1						
3.						1		1					
4.						1		2					
5.					1		3		1				
6.					1		4		3				
7.				1		5		6		1			
8.				1		6		10		4			
9.			1		7		15		10		1		
10.			1		8		21		20		5		
11.		1		9		28		35		15		1	
12.		1		10		36		56		35		6	
13.	1		11		45		84		70		21		1
14.	1		12		55		120		126		56		7

Figure: from Falcón and Plaza based on $F_n(x)$, Hogatt's polynomial

Pascal triangles in $G_n(x)$: $k = l = 1$

$G_0(x)$							1						
$G_1(x)$						1		-1					
$G_2(x)$						1		-2					
$G_3(x)$					1		-1		-1				
$G_4(x)$					1		0		-3				
$G_5(x)$				1		1		-4		1			
$G_6(x)$				1		2		-4		-4			
$G_7(x)$			1		3		-3		-8		-1		
$G_8(x)$			1		4		-1		-12		-5		
$G_9(x)$		1		5		2		-15		-13		-1	
$G_{10}(x)$		1		6		6		-16		-25		-6	
$G_{11}(x)$	1		7		11		-14		-40		-19		-1
$G_{12}(x)$	1		8		17		-8		-56		-44		-7

Pascal triangles in $G_n(x)$: $k = l = 1$

$G_0(x)$							1						
$G_1(x)$						1		-1					
$G_2(x)$						1		-2					
$G_3(x)$					1		-1		-1				
$G_4(x)$					1		0		-3				
$G_5(x)$				1		1		-4		1			
$G_6(x)$				1		2		-4		-4			
$G_7(x)$			1		3		-3		-8		-1		
$G_8(x)$			1		4		-1		-12		-5		
$G_9(x)$		1		5		2		-15		-13		-1	
$G_{10}(x)$		1		6		6		-16		-25		-6	
$G_{11}(x)$	1		7		11		-14		-40		-19		-1
$G_{12}(x)$	1		8		17		-8		-56		-44		-7

Pascal triangles in $G_n(x)$: $k = l = 2$

$G_0(x)$						1					
$G_1(x)$					1		-1				
$G_2(x)$					1		-2				
$G_3(x)$				1	-2		1	-1			
$G_4(x)$				1	-2		2	-3			
$G_5(x)$			1	-2	3		-5	1	-1		
$G_6(x)$			1	-2	4		-7	3	-4		
$G_7(x)$		1	-2	5	-9		6	-9	1	-1	
$G_8(x)$		1	-2	6	-11		10	-16	4	-5	
$G_9(x)$	1	-2	7	-13	15		-25	10	-14	1	-1
$G_{10}(x)$	1	-2	8	-15	21		-36	20	-30	5	-6

Observe how elements in each row increase by 2

Pascal triangles in $G_n(x)$: $k = l = 2$

$G_0(x)$					1					
$G_1(x)$					1	-1				
$G_2(x)$					1	-2				
$G_3(x)$				1	-2	1	-1			
$G_4(x)$				1	-2	2	-3			
$G_5(x)$			1	-2	3	-5	1	-1		
$G_6(x)$			1	-2	4	-7	3	-4		
$G_7(x)$		1	-2	5	-9	6	-9	1	-1	
$G_8(x)$		1	-2	6	-11	10	-16	4	-5	
$G_9(x)$	1	-2	7	-13	15	-25	10	-14	1	-1
$G_{10}(x)$	1	-2	8	-15	21	-36	20	-30	5	-6

Observe how elements in each row increase by 2

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Pascal triangles in $G_n(x)$: $k = 1, l = 2$?

$G_0(x)$	$1x^0$	
$G_1(x)$	$1x^1$	$-1x^0$
$G_2(x)$	$0x^2$	$-1x^1$
$G_3(x)$	$1x^3$	$-2x^2$
$G_4(x)$	$1x^4$	$-3x^3$
$G_5(x)$	$2x^5$	$-5x^4$
$G_6(x)$	$3x^6$	$-8x^5$
$G_7(x)$	$5x^7$	$-13x^6$
$G_8(x)$	$8x^8$	$-21x^7$
$G_9(x)$	$13x^9$	$-34x^8$
$G_{10}(x)$	$21x^{10}$	$-55x^9$
$G_{11}(x)$	$34x^{11}$	$-89x^{10}$

$$G_n(x) = F_{n-2}x^n - F_nx^{n-1}$$

General matrix representations of $G_n(x)$

Formula One:

$$\begin{bmatrix} G_{n+2}(x) & G_{n+1}(x) \\ G_{n+1}(x) & G_n(x) \end{bmatrix} = \begin{bmatrix} x^k & x^l \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} G_3(x) & G_2(x) \\ G_2(x) & G_1(x) \end{bmatrix}$$

Formula Two:

$$\begin{bmatrix} G_{n+4} & G_{n+3} & G_{n+2} \\ G_{n+3} & G_{n+2} & G_{n+1} \\ G_{n+2} & G_{n+1} & G_n \end{bmatrix} = \begin{bmatrix} x^k & x^l & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} G_5 & G_4 & G_3 \\ G_4 & G_3 & G_2 \\ G_3 & G_2 & G_1 \end{bmatrix}$$

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Binet forms of $G_n(x)$ sequences

Sequence for $x = 2$:

G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_n
-1	1	0	2	4	12	32	88	240	656	

For Initial Conditions $G_0 = -1$ and $G_1 = 1$:

$$G_n = (\frac{\sqrt{3}}{3} - \frac{1}{2})(1 + \sqrt{3})^n + (-\frac{\sqrt{3}}{3} - \frac{1}{2})(1 - \sqrt{3})^n$$

Example for $n = 6$:

$$G_6 = (\frac{\sqrt{3}}{3} - \frac{1}{2})(1 + \sqrt{3})^6 + (-\frac{\sqrt{3}}{3} - \frac{1}{2})(1 - \sqrt{3})^6 = 32$$

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Binet form to find $G_n(x)$ itself

The following formula produces the exact result for $G_n(x)$:

$$G_n(x) = \frac{\sqrt{x^2+4x} - 3x + 2}{-2\sqrt{x^2+4x}} \left(\frac{x + \sqrt{x^2+4x}}{2} \right)^n + \frac{\sqrt{x^2+4x} + 3x - 2}{-2\sqrt{x^2+4x}} \left(\frac{x - \sqrt{x^2+4x}}{2} \right)^n$$

Shifted Fibonacci numbers

We observed that when $x = 1$, we always yielded the negative Fibonacci numbers $\forall\ n \in \mathbb{N}$ where $n > 0$.

G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_n
$-F_0$	$-F_1$	$-F_2$	$-F_3$	$-F_4$	$-F_5$	$-F_6$	$-F_7$	$-F_{n-1}$
0	-1	-1	-2	-3	-5	-8	-13	

We observed this pattern persisted for cases where $k = l$.

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More shifted Fibonacci numbers

Say $k = 2$, $l = 4$, and $x = -1$. Then we get:

G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_n
$-F_2$	$-F_3$	$-F_4$	$-F_5$	$-F_6$	$-F_7$	$-F_8$	$-F_9$	$-F_{n+2}$
-1	-2	-3	-5	-8	-13	-21	-34	

The pattern seems to hold whenever **k and l are even, and $x = -1$.**

Shifted Lucas numbers

Take this example where $k = 1$, $l = 2$ and $x = -1$:

G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_n
$-L_0$	L_1	$-L_2$	L_3	$-L_4$	L_5	$-L_6$	L_7	$(-1)^n * L_{n-1}$
-2	1	-3	4	-7	11	-18	29	

The pattern seems to hold when k **is odd,** l **is even, and** $x = -1$**.**

Cyclical sequences - 3 terms

What if we plug in $x = -1$ instead?:

G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_n
-1	-2	3	-1	-2	3	-1	-2	3	$G_{3n} = -1$ $G_{3n+1} = -2$ $G_{3n+2} = 3$

This seems to hold whenever k **and** l **are odd and** $x = -1$**.**

Cyclical sequences - 6 terms

Now let's look at the opposite, $k = 2$ and $l = 1$ for $x = -1$:

G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_n
-1	-2	-1	1	2	1	-1	-2	-1	$G_{6n} = -1$ $G_{6n+1} = -2$ $G_{6n+2} = -1$ $G_{6n+3} = 1$ $G_{6n+4} = 2$ $G_{6n+5} = 1$

The pattern seems to hold so long as k **is even,** l **is odd, and** $x = -1$.

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Ratios between $G_n(x)$ sequences

To explain our next results, we will introduce a new notation.

$$\lim_{n\to\infty} \frac{G_{n+1}(x)}{G_n(x)} = ???$$

Continued fraction notation - linear form

Number	Also known as	As a continued fraction
1.5	$\frac{3}{2}$	$[1; 2]$
$2.\overline{66}$	$\frac{8}{3}$	$[2; 1, 2]$
$3.1415\ldots$	π	$[3; 7, 15, 1, 292, 1, 1, \ldots]$
1.4141...	$\sqrt{2}$	$[1; \overline{2}]$

Continued fraction example - Φ

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

In linear form Φ is expressed as

$$[1; 1, 1, 1, 1, 1, 1, \ldots] \iff [1; \overline{1}]$$

Results for $k = l = 1$ - golden numbers

x	$\lim_{n\to\infty} \frac{G_{n+1}(x)}{G_n(x)}$	Continued fraction form
1	1.618034	$[1; \overline{1}]$
2	2.732051	$[2; \overline{1, 2}]$
3	3.791288	$[3; \overline{1, 3}]$
4	4.828427	$[4; \overline{1, 4}]$
5	5.854102	$[5; \overline{1, 5}]$
6	6.872983	$[6; \overline{1, 6}]$
7	7.887482	$[7; \overline{1, 7}]$

Results for $k = l = 2$

x	$\lim_{n\to\infty} \frac{G_{n+1}(x)}{G_n(x)}$	Continued fraction form
1	1.618034	$[1; \overline{1}]$
2	4.828427	$[4; \overline{1, 4}]$
3	9.908326	$[9; \overline{1, 9}]$
4	16.94427	$[16; \overline{1, 16}]$
5	25.96291	$[25; \overline{1, 25}]$
6	36.97366	$[36; \overline{1, 36}]$
7	49.98038	$[49; \overline{1, 49}]$

Conjecture $\forall\, x \in \mathbb{N} : k = l$

$$\lim_{n\to\infty} \frac{G_{n+1}(x)}{G_n(x)} = [\,x^k; \overline{1, x^k}]$$

$$= x^k + \cfrac{1}{1 + \cfrac{1}{x^k + \cfrac{1}{1 + \cfrac{1}{x^k + \cfrac{1}{\dots}}}}}$$

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Results for $k = 2$, $l = 1$

x	$\lim_{n\to\infty} \frac{G_{n+1}(x)}{G_n(x)}$	Continued fraction form
1	1.618034	$[1; \overline{1}]$
2	4.449489	$[4; \overline{2, 4}]$
3	9.321825	$[9; \overline{3, 9}]$
4	16.24621	$[16; \overline{4, 16}]$
5	25.19842	$[25; \overline{5, 25}]$
6	36.16590	$[36; \overline{6, 36}]$
7	49.14244	$[49; \overline{7, 49}]$

Conjecture $\forall x \in \mathbb{N} : k \neq l, k > l$

$$\lim_{n\to\infty} \frac{G_{n+1}(x)}{G_n(x)} = [x^k; \overline{x^l, x^k}]$$

$$= x^k + \cfrac{1}{x^l + \cfrac{1}{x^k + \cfrac{1}{x^l + \cfrac{1}{x^k + \cfrac{1}{\dots}}}}}$$

Results for $k = 1, l = 2$

x	$\lim_{n\to\infty} \frac{G_{n+1}(x)}{G_n(x)}$	Alternate form
1	1.618034	$1 * \Phi$
2	3.236068	$2 * \Phi$
3	4.854098	$3 * \Phi$
4	6.472134	$4 * \Phi$
5	8.090168	$5 * \Phi$
6	9.708202	$6 * \Phi$
7	11.32623	$7 * \Phi$

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Conjecture $\forall x \in \mathbb{N} : k = 1, l = 2$

$$\lim_{n\to\infty} \frac{G_{n+1}(x)}{G_n(x)} = x * \Phi$$

These results are also conjectured about $\forall\ x \in \mathbb{Z}$.

Other interesting results for negative x will continue to be researched, observed and conjectured about further.

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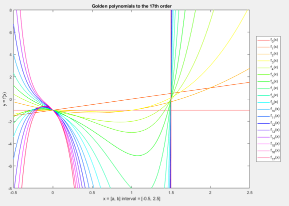
Utility of the generalized $G_n(x)$



Generalizing $G_n(x)$ allows us to make conclusions about its max root g_n. Similar to Moore we

- looked for a pattern - graphing and calculating $G_n(x)$
- proved existence of roots - using sequences to support the claim
- identified whether g_n was increasing, decreasing - or both

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Root for $k = 2$ and $l = 1$

This is really cool - as $n \to \infty$ we see that the limit of the max root g_n of $G_n(x)$ approaches this quantity:

$$\frac{1}{1 - 2 * \sin\left(\frac{\pi}{18}\right)} \iff \frac{1}{p_c(\text{honeycomb bond})}$$

- This interesting quantity in the denominator is an exact quantity of what's referred to as p_c(honeycomb bond).
- This is a constant related to the field of percolation theory
- Determining an exact expression for other percolation thresholds, including of the square site percolation, remains an open problem - one which could be studied further with continued research of recursive polynomials - and a nice hot cup of coffee!

Root for $k = 1$ and $l = 2$

This is really cool - as $n \to \infty$ we see that the limit of the max root g_n of $G_n(x)$ approaches this quantity - though convergence is not as obvious:

$$\Phi + 2 \iff \frac{3+\sqrt{5}}{2} \iff [2; \overline{1}] \iff 2 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}}$$

- For fun we checked a couple of other cases keeping $k = 1$ but increasing l, unfortunately the pattern did not continue.
- This can be explained by the unique quality of the $k = 1$ and $l = 2$ case where there are only ever 2 terms.
- Increasing l creates more terms, which in turn, creates more roots that breaks the identified pattern

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Concluding remarks

With continued research, we seek to:

- Generalize the behavior of sequences and the ratio between sequences for a recursive polynomial, based on the parity of k and l
- Extend matrix results to 4 x 4, 5 x 5, and up to n x n matrices
- Explore the existence of Pascal-3, Pascal-4, and up to Pascal-m type triangles for various k and l
- Characterize complex roots and the range in which the entirety of roots of G_n can be found

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