Bayesian Estimators for the Reproduction Number of Epidemics

Elijah Hight
Advisor:Dr. George Yanev

COS 2020, November 20, 2020

Overview

Borel-Tanner Distribution

The probability function of Borel-Tanner (BT) distribution is

$$p_r(x;\theta) = c_r(x)\theta^{x-r}e^{-x\theta} \qquad x = r, r+1, \ldots, \tag{1}$$

where $0 < \theta < 1$, $r \in \mathbb{Z}^+$ and $c_r(x) = \frac{rx^{x-r-1}}{(x-r)!}$. (1) has a mean $\frac{r}{1-\theta}$ and variance $\frac{r\theta}{(1-\theta)^3}$

Borel-Tanner Distribution

The probability function of Borel-Tanner (BT) distribution is

$$p_r(x;\theta) = c_r(x)\theta^{x-r}e^{-x\theta} \qquad x = r, r+1, \ldots, \tag{1}$$

where $0 < \theta < 1$, $r \in \mathbb{Z}^+$ and $c_r(x) = \frac{rx^{x-r-1}}{(x-r)!}$. (1) has a mean $\frac{r}{1-\theta}$ and variance $\frac{r\theta}{(1-\theta)^3}$

- Derived as the distribution of the number of customers served in a single queuing process

Borel-Tanner Distribution

The probability function of Borel-Tanner (BT) distribution is

$$p_r(x;\theta) = c_r(x)\theta^{x-r}e^{-x\theta} \qquad x = r, r+1, \ldots, \tag{1}$$

where $0 < \theta < 1$, $r \in \mathbb{Z}^+$ and $c_r(x) = \frac{rx^{x-r-1}}{(x-r)!}$. (1) has a mean $\frac{r}{1-\theta}$ and variance $\frac{r\theta}{(1-\theta)^3}$

- Derived as the distribution of the number of customers served in a single queuing process
- In 1942, Emil Borel introduced the distribution for $r = 1$

Borel-Tanner Distribution

The probability function of Borel-Tanner (BT) distribution is

$$p_r(x;\theta) = c_r(x)\theta^{x-r}e^{-x\theta} \qquad x = r, r+1, \ldots, \tag{1}$$

where $0 < \theta < 1$, $r \in \mathbb{Z}^+$ and $c_r(x) = \frac{rx^{x-r-1}}{(x-r)!}$. (1) has a mean $\frac{r}{1-\theta}$ and variance $\frac{r\theta}{(1-\theta)^3}$

- Derived as the distribution of the number of customers served in a single queuing process
- In 1942, Emil Borel introduced the distribution for $r = 1$
- In 1953, Tanner generalized for $r \in \mathbb{Z}^+$

Applications

Queueing Models

The number of customers served in a busy period of a single-server queuing process, started with r customers, assuming Poisson arrivals and constant service time.

Branching Processes (Epidemic Models)

If the number of offspring of an individual is Poisson distributed, then the total progeny is a BT random variable.

Others

Coalescence models, highway traffic flows, propagation of internet viruses, cascading failures of energy systems, herd size in finance modeling, epidemic infections modeled by a branching process.

LINEX Loss Function

For this research we are interested in estimating the parameter θ, the reproductive number, or the average number of secondary infections caused by a host. It is thus important to consider the severity of under/over estimating θ.

Varian (1975) first defined the linear exponential (linex) loss function as the following:

LINEX Loss Function

For this research we are interested in estimating the parameter θ, the reproductive number, or the average number of secondary infections caused by a host. It is thus important to consider the severity of under/over estimating θ.

Varian (1975) first defined the linear exponential (linex) loss function as the following:

Definition

Let a be an estimate of the parameter θ, then the linex loss function

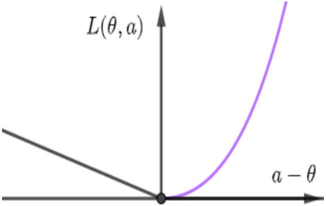
$$L(\theta, a) = e^{b(a-\theta)} - b(a-\theta) - 1, \quad b \neq 0$$

LINEX Loss Function

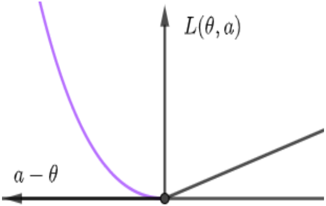
$$L(\theta, a) = e^{b(a-\theta)} - b(a-\theta) - 1, \quad b \neq 0$$

It's important to note the asymmetry of the linex error above.

- For $b > 0$, $L(\theta, a)$ penalizes overestimation more than underestimation



- For $b < 0$, $L(\theta, a)$ penalizes underestimation more than overestimation



Classical Bayes Estimators

Let (X, Θ) be a pair of r.v., where X corresponds to the observable variable and Θ corresponds to the unknown parameter.

Suppose the parameter θ is a realization of $0 < \Theta < 1$

Given Θ has a prior distribution G, the Bayes estimate is

$$\varphi_G(x) = -\frac{1}{b} \ln E \left[e^{-b\Theta} | X = x \right]$$

Assuming G is $Uni(m, n)$ for θ and we can then reduce the above equation to the following

$$\Rightarrow \varphi_G(x) = -\frac{1}{b} \ln \frac{\psi_G(x, b)}{q_G(x)} = -\frac{1}{b} \ln \frac{\int_m^n e^{-b\theta} \theta^{x-r} e^{-x\theta} \, d\theta}{\int_m^n \theta^{x-r} e^{-x\theta} \, d\theta}$$

Empirical Bayes Approach

Many times it is reasonable to assume that there exists a prior distribution, which however, is unknown. The empirical Bayes approach offers a solution when the experiment under investigation has been preceded by a series of comparable experiments. Then the observations gathered from the preceding experiments can be used to obtain information about the prior.

Empirical Bayes Estimators for BT

Liang (2009) constructed an EB estimator for θ at x as follows: For each integer $x = r, r+1, \ldots$ and $k = 1, 2, \ldots$ and each $j = 1, \ldots, n$, let

$$\psi_{nj0}(x) = I\left[X_j = x\right]/c_r(x)$$

$$\psi_{njk}(x) = c_k(X_j - x)I[X_j \geq x + k]/c_r(X_j)$$

$$\psi_{nj}(x, b) = \sum_{k=0}^{\infty} \frac{(-b)^k}{k!}\psi_{njk}(x)$$

Define

$$\psi_n(x, b) = \frac{1}{n} \sum_{j=1}^{n} \psi_{nj}(x, b)$$

$$q_n(x) = \frac{1}{n} \sum_{j=1}^{n} \psi_{nj0}(x)$$

Lastly recall that $\varphi_G(x) = -\frac{1}{b} \ln \frac{\psi_G(x,b)}{q_G(x)}$ Then, for each $x = r, r+1, \ldots,$ define the EB estimator for θ at x as

$$\varphi_n(x) = -\frac{1}{b} \ln \left[(m_1(b) \vee \frac{\psi_n(x, b)}{q_n(x)}) \wedge m_2(b) \right],$$

where $m_1(b) = \min(e^{-b}, 1)$ and $m_2(b) = \max(e^{-b}, 1)$

Monotonization of the EB Estimate

The following property can be shown:

Proposition 1 - Soltero (2018)

The Borel-Tanner distribution has a monotone likelihood ratio

$$q(x) = \frac{p_r(x \mid \theta_2)}{p_r(x \mid \theta_1)}$$

which is increasing with respect to x when $0 < \theta_1 < \theta_2 < 1$

Hence, monotonicity is a desirable property for the EB estimator however, as we will see late, this is not the case for the EB estimator.

Randomization

Randomization procedures are designed to "control" bias as much as possible. Estimators for discrete distributions, BT for instance, with monotone likelihood ratio, can be made monotone using a procedure developed by Houwalingen (1977). Our procedure is as follows:

For $a \in (0, 1)$ the simple randomized version of the estimator $\varphi_n(x)$ is represented by:

$$D(a \mid x) = \begin{cases} 0 & if\, \varphi_n(x) > a \\ 1 & if\, \varphi_n(x) \leq a \end{cases}$$

Randomization (cont.)

$D(a \mid x)$ is the probability that an estimate $\varphi_n(x) \leq a$ is selected given $X = x$. We define for $a \in (0, 1)$

$$\alpha(a) = E[D(a \mid x)] = \sum_{x:\ \varphi_n(x) \leq a} p_r(x \mid a)$$

And assume

$$P(x \mid \theta) = \sum_{k=r}^{x} p_r(k \mid \theta) \ for \ x \geq r; \ assume \ P(r-1 \mid \theta) = 0$$

Randomization (cont.)

$D(a \mid x)$ is the probability that an estimate $\varphi_n(x) \leq a$ is selected given $X = x$. We define for $a \in (0, 1)$

$$\alpha(a) = E[D(a \mid x)] = \sum_{x:\ \varphi_n(x) \leq a} p_r(x \mid a)$$

And assume

$$P(x \mid \theta) = \sum_{k=r}^{x} p_r(k \mid \theta) \ \textit{for } x \geq r; \ \textit{assume } P(r-1 \mid \theta) = 0$$

Thus the randomized monotone estimator is defined as follows:

$$D^*(a \mid x) = \begin{cases} 0 & \textit{if } \alpha(a) < P(x-1 \mid a) \\ \frac{\alpha(a) - P(x-1|a)}{P(x|a) - P(x-1|a)} & \textit{if } P(x-1 \mid a) \leq \alpha(a) \leq P(x \mid a) \\ 1 & \textit{if } P(x \mid a) < \alpha(a) \end{cases}$$

Numerical Study

Let X be a discrete random variable following a BT distribution with a $Uni(0.5, 0.8)$ prior G for Θ and let $r = 3,\ b = 1$ and $x = 3, 4, \ldots, 20$. Then

$$\textit{Bayesian Estimator}: \varphi_G(x) = -\ln \frac{\int_{0.5}^{0.8} e^{-\theta}\theta^{x-3}e^{-x\theta}\ d\theta}{\int_{0.5}^{0.8} \theta^{x-3}e^{-x\theta}\ d\theta}$$

We then compute $\varphi_G(x)$ for each value of $x = 3, 4, \ldots, 20$ using R statistical software

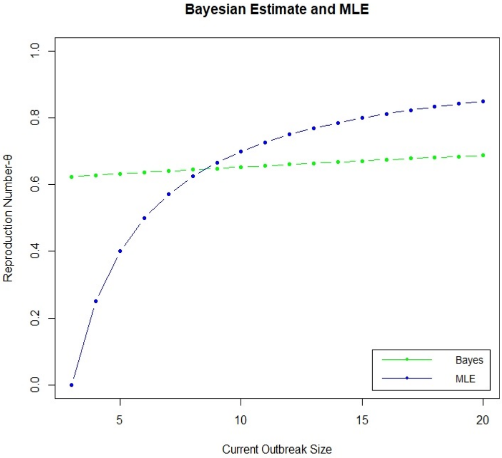
Computing the EB Estimate and Monotone Estimate

Next we assume independent identically distributed of the random pair (X, Θ), where Θ has $Uni(0.5, 0.8)$ prior G and let $r = 3,\ b = 1$ and $n = 20.$ We then want to compute

$$\psi_{20}(x, 1) = \frac{1}{20} \sum_{j=1}^{20} \psi_{20,j}(x, 1)$$

$$q_{20}(x) = \frac{1}{20} \sum_{j=1}^{20} \psi_{20,j0}(x)$$

Numerical Results: r=3, b=1, n=20



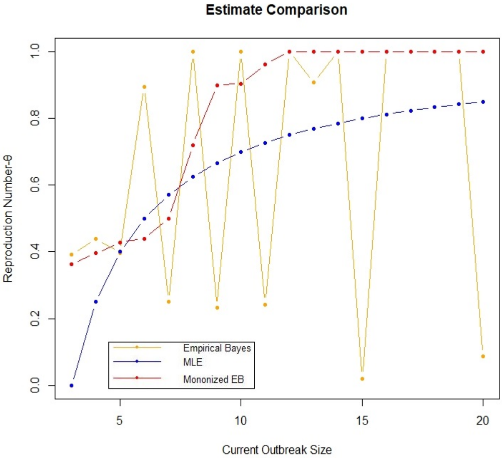


Figure: Estimate comparison for one realization at n=20

Regret Risks

Definition

The difference $R(\hat{\theta})$ between the Bayes risk and minimum Bayes risk of any estimator $\hat{\theta}$ is called the Regret Risk

n	$R(\theta_{MLE})$	$\bar{R}(\varphi_n)$	$\bar{R}(\varphi_n^*)$
20	0.03828749	0.06126433	0.03592903
40	0.03828749	0.04889400	0.02745336
60	0.03828749	0.04375859	0.02419265
80	0.03828749	0.04346781	0.02398901

Figure: Note that the Regret of the monotonized Estimator φ_n^* is the smallest implying it is a better estimator than the MLE and EB

Effects of b on the Regret Risks

$$L(\theta, a) = e^{b(a-\theta)} - b(a-\theta) - 1, \quad b \neq 0$$

We wanted to investigate how changing the value of b effects the performance of the estimates. (i.e. Regret Risk) Below are the results:

b	*Regret MLE*	*Regret EB*	*Regret MonoEB*
-2	0.275686467	0.416765871	0.059577410
-1.5	0.138446310	0.217465760	0.036544862
-1	0.055298961	0.091491834	0.017739986
-.5	0.012502722	0.022034358	0.004820517
.5	0.010406704	0.014010409	0.008287252
1	0.038287493	0.061264330	0.035929034
1.5	0.079634630	0.149197955	0.087793862
2	0.131492256	0.290226135	0.175857634

Figure: Note for $b < 0$, the monotonized EB has lower regret than when $b > 0$; this is ideal as we prefer a framework where we penalize underestimation more than overestimation

Conclusions and Future Objectives

- The current objective is to investigate the risk ratios between the monotonized estimates and the MLE/EB estimates. Then determine for what value(s) of b is our monotonized estimate admissable.

- Consider the case where the initial infected cases r, is Poisson random distributed and apply the same procedures when computing the estimates

References

Houwelingen, J.C. (1977)
Monotonizing emprical Bayes estimators for a class of discrete distributions with monotone likelihood ratio. .
Statistica Neerlandica 31, 95 – 104.

Liang, T. (2009)
Empirical Bayes estimation for Borel-Tanner distributions.
Stat. and Probab. Lett. 79, 2212 – 2219.

Soltero, C. and Yanev, G. (2018)
On Monotone Empirical Bayes Estimation for Borel-Tanner Distribution.
submitted

Varian, Hal R. (1975)
A Bayesian Approach to Real Estate Assessment.
Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage
195–208

Yanev, G.P. and Colson, R. (2017)
Monotone empirical Bayes estimators for the reproduction number in Borel-Tanner distribution.
Pliska 27, 115-122.