Boundary Feedback Control of the 3D Navier-Stokes Equations

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3D Navier-Stokes Equations

$$\vec{W}_t - \nu \Delta \vec{W} + \left(\vec{W} \cdot \nabla\right) \vec{W} + \nabla P = 0, \quad \mathbf{x} \in \Omega, \quad t > 0$$
$$\operatorname{div} \vec{W} = 0$$

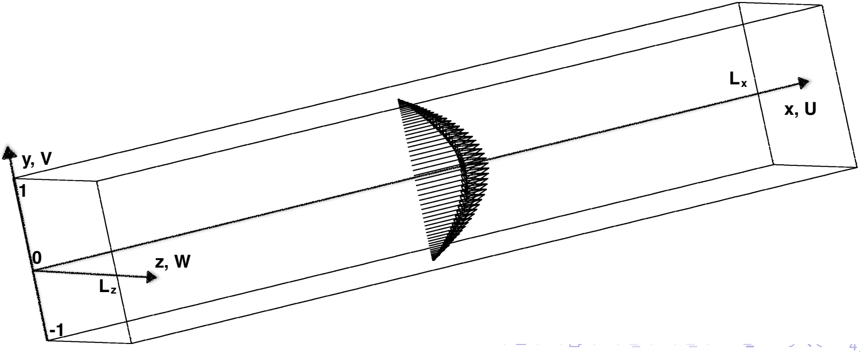
- spatial variable: $\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3$, time variable: t
- viscosity: $\nu > 0$
- flow velocity:
  $\vec{W} = \vec{W}(x, y, z, t) = [U(x, y, z, t), V(x, y, z, t), W(x, y, z, t)]^T$
- pressure: $P = P(x, y, z, t)$
- Divergence-free, incompressible flow: $\operatorname{div} \vec{W} = U_x + V_y + W_z = 0$

History and Applications

- In the early 1800s, the equation was derived by G. G. Stokes and N. Navier, as an extension to Euler's equation.
- Euler's equation describes the flow of incompressible and frictionless fluid, while the Navier-Stokes equation includes the viscosity of the flow.
- The equation represents the system of partial differential equations that describe the motion of a fluid and gas flow in space and time.
- It is important in many applications such as a flow in a pipe, airflow around an airplane wing, ocean currents, and weather.
- Controlling it is needed in practical applications such as reducing turbulence, drag, enhancing mixing.
- Despite the practical importance, the existence and uniqueness of solutions to the equations is an open question.

3D Channel Flow

- Simplified geometry for theory and computation
- $\vec{W} = \vec{W}(x, y, z, t) = [U(x, y, z, t), V(x, y, z, t), W(x, y, z, t)]^T$
- Wall at the top and bottom $y = -1, +1$
- Flow periodicity in the x and z directions
- Pressure drop from $x = 0$ inlet to $x = L_x$ outlet



Periodic Boundary Conditions in x and z directions

To simplify calculations and to speed up computations.

$$U(0,y,z,t) = U(L_x,y,z,t), \qquad W(0,y,z,t) = W(L_x,y,z,t)$$

$$V(0,y,z,t) = V(L_x,y,z,t), \qquad U(x,y,0,t) = U(x,y,L_z,t)$$

$$V(x,y,0,t) = V(x,y,L_z,t), \qquad W(x,y,0,t) = W(x,y,L_z,t)$$

$$V_x(0,y,z,t) = V_x(L_x,y,z,t), \qquad V_z(x,y,0,t) = V_z(x,y,L_z,t)$$

$$U_z(x,y,0,t) = U_z(x,y,L_z,t), \qquad W_x(0,y,z,t) = W_x(L_x,y,z,t)$$

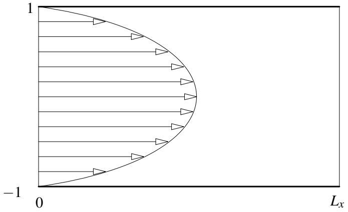
$$P(0,y,z,t) = P(L_x,y,z,t) + aL_x, \quad P(x,y,0,t) = P(x,y,L_z,t)$$

The flow is pushed trough the channel by constant pressure drop a.

Steady State Solution (Poiseuille Flow) (Laminar Flow)

Time independent solution: $\overline{\mathbf{W}} = \left(\overline{U}, \overline{V}, \overline{W}\right)^T$, where

$$\overline{U}\left(y\right) = \frac{a}{2\nu}\left(1 - y^2\right)$$
$$\overline{V} = \overline{W} = 0$$
$$\overline{P} = -ax + b$$



This ideal parabolic flow profile (no turbulence, lowest possible drag at the walls) does not happen in practice at high flow velocities.

Control Objectives

- The goal is to introduce special boundary conditions (feedback control) in order to stabilize the laminar flow:

$$\vec{W}(x,y,z,t) \xrightarrow{t\to\infty} \overline{\vec{W}}(x,y,z) \text{ for all } x,y,z$$

- In terms of perturbation variable:

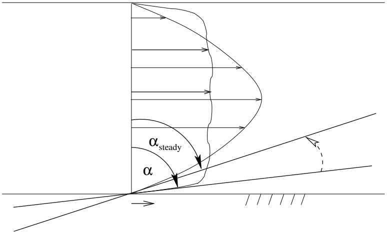
$$\vec{w}(x,y,z,t) = \vec{W}(x,y,z,t) - \overline{\vec{W}}(x,y,z) \xrightarrow{t\to\infty} 0$$

- The control law is derived using Lyapunov's method on the L^2–norm of the perturbation variable (perturbation energy).

$$E(t) = \|\vec{w}\| = \sqrt{\iiint_V (u^2 + v^2 + w^2)\, dx\, dy\, dz} \xrightarrow{t\to\infty} 0$$

$$U(x,-1,z,t) = k\left[U_y(x,-1,z,t) - \frac{a}{\nu}\right]$$
$$U(x,1,z,t) = -k\left[U_y(x,1,z,t) + \frac{a}{\nu}\right]$$
$$U_{\text{control at wall}} = \text{control gain } k \times (\text{error in shear stress})$$

- The shear stress is adjusted to that of the steady state profile
- Wall–tangential (streamwise) velocity actuation
- Other velocity components are kept at zero: $V_{\text{wall}} = W_{\text{wall}} = 0$



$$U_y = \tan\alpha$$
$$\frac{a}{\nu} = \tan\alpha_{\text{steady}}$$

Main Theorem

Theorem: *If conditions* $\sigma = \frac{\nu}{4} - \frac{a}{2\nu} > 0$, *(i.e.,* $\nu > \sqrt{2a}$*),* $0 < k < 1$ *are satisfied, then the steady state solution is globally exponentially stable in the* L_2*–sense, i.e., in terms of the perturbation variables:*

$$\|\vec{w}\,(t)\,\| \le \|\vec{w}_0\| e^{-\sigma t}$$

for all $t > 0$ *and for all initial conditions* $\vec{w}_0$.

Remarks:

- Conservative theoretical result, the viscosity ν has to be large: $\nu > \sqrt{2a}$, where a is the constant pressure drop between the two ends of the channel.
- Numerical simulations show effectiveness of the control for smaller viscosity/larger Reynolds number.

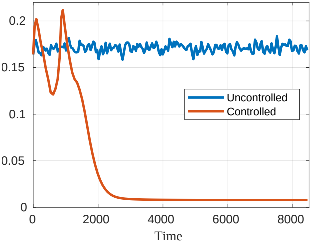
Numerical Simulations

- Hybrid Fourier pseudospectral–finite difference discretization and fractional step technique based on a hybrid Runge–Kutta/Crank–Nicolson finite discretization (Bewley, Moin, & Temam)
- Boundary control (Balogh): implicit three–point end–point formula
- MPI parallel Fortran code, running on the Lonestar5 Cray XC40 Cluster of the Texas Advanced Computing Center
- Reynolds number $Re_\tau = \dfrac{U_{\text{centerline}}}{\nu} \approx 2,000$
- Channel dimensions $6\pi \times 2 \times 2\pi$
- Resolution $128 \times 200 \times 64$

Scientific Visualization of Computational Results

- The visualization of the computational results has been done using the ParaView open-source, multi-platform data analysis and visualization application.

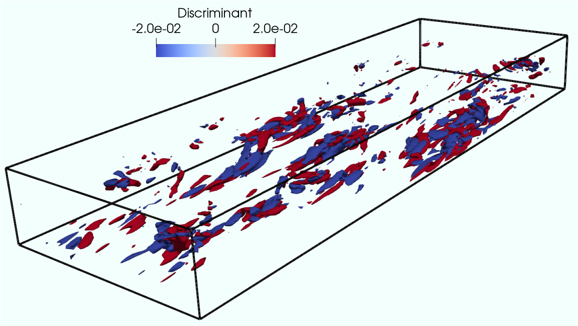
L_2 Perturbation Energy



- The uncontrolled perturbation energy oscillates around a constant value due to turbulence (statistically steady flow).
- The control initially creates a large-amplitude oscillation (nonlinear phase).
- The exponential decay from around $t = 1,000$ indicates linear behaviour, like the solution to a linear ode of the form $y'(t) = -y(t)$.

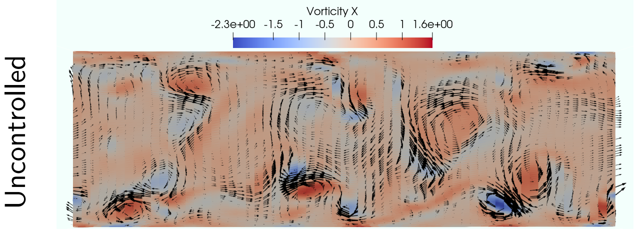
Discriminant

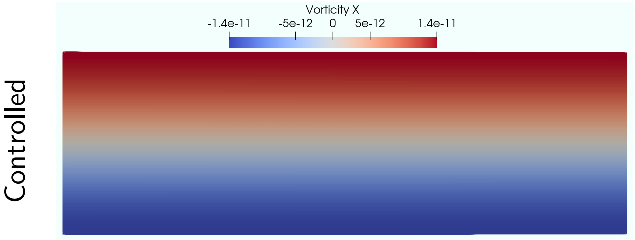
- Isosurfaces of the discriminant of the velocity gradient tensor for the uncontrolled flow (shown only at the bottom wall).
- Turbulent structures arising from the boundary layer at the walls.
- Controlled case shows no such thing.



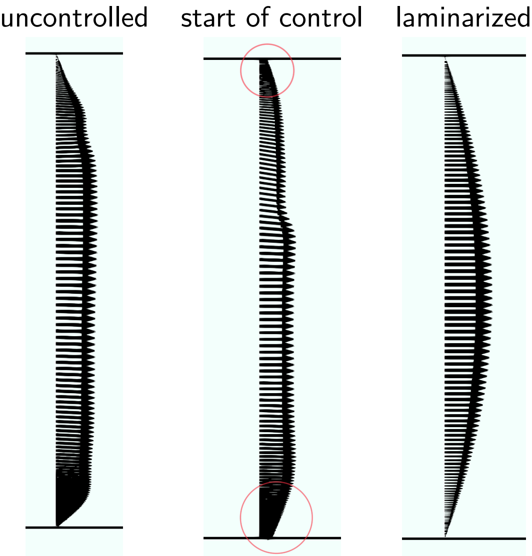
Spanwise Velocity and Streamwise Vorticity from u, v, w

- Spanwise velocity: (v, w) showing vortex structures
- Streamwise vorticity (x-component of vorticity): $\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}$





Velocity Profiles



Summary and Future Plans

Summary

- Our boundary feedback control law stabilizes the ideal parabolic flow profile of the 3D Navier-Stokes equation.
- Theory gives limited results (low Reynolds number)
- By visualizing different flow quantities we saw that numerical simulations extend the results to high Reynolds numbers ($Re = 2,000$).

Future Plans

- Prove well-posedness (existence and uniqueness) of the solution (for low Reynolds numbers).
- Examine the drag and other flow quantities in simulations.
- Find the highest Reynolds number for which the control works.
- Check for high Reynolds numbers if turbulence is reduced.

References

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