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Continuity for a generalized cross-coupled Camassa–Holm system with waltzing peakons and higher-order nonlinearities ☆

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ABSTRACT

In this paper, we consider the well-posedness problem in the sense of Hadamard, non-uniform dependence, and Hölder continuity of the data-to-solution map for a generalized cross-coupled Camassa–Holm system with waltzing peakons on both the periodic and the non-periodic case. In light of a Galerkin-type approximation scheme, the system is shown well-posed in the Sobolev spaces $H^s \times H^s, s > 5/2$ in the sense of Hadamard, that is, the data-to-solution map is continuous. However, the solution map is not uniformly continuous. Furthermore, we prove the Hölder continuity in the $H^r \times H^r$ topology when $0 \le r < s$ with Hölder exponent α depending on both s and r.

1. Introduction

In this paper, we deal with the well-posedness problem in the sense of Hadamard, non-uniform dependence, and Hölder continuity of the data-to-solution map for the following generalized cross-coupled Camassa–Holm system

$$\begin{cases} m_t + v^p m_x + a v^{p-1} v_x m = 0, & t \in \mathbb{R}, \\ n_t + u^q n_x + b u^{q-1} u_x n = 0, & t \in \mathbb{R}, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & t = 0, \end{cases} \tag{1.1}$$

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where $m = u - u_{xx},\ n = v - v_{xx}$, $p, q \in \mathbb{Z}^+$, and a, b are two constant parameters. Obviously, Eq. (1.1) has nonlinearities of degree $\max\{p+1, q+1\}$.

The water wave and fluid dynamics have been attracting much attention during last few decades due to various mathematical problems and nonlinear physics phenomena interfered [1]. Since the raw water wave governing equations are nearly intractable, the request for suitably simplified model equations was initiated at the early stage of hydrodynamics development. Until the early twentieth century, the study of water waves was confined almost exclusively to the linear theory. Due to the linearization approach losing some important properties, people usually propose some nonlinear models to explain practical behaviors like breaking waves and solitary waves [2]. The most prominent example is the following family of nonlinear dispersive partial differential equations

$$u_t - \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \tag{1.2}$$

where γ, α, c_1, c_2, and c_3 are real constants. By the Painlevé analysis method in [3–5], there are only three asymptotically integrable members in this family: the KdV equation (Eq. (1.2) with $\alpha = c_1 = c_2 = 0$, cf. [6]), the Camassa–Holm equation (Eq. (1.1) with $u = v, p = q = 1, a = b = 2$, cf. [7–9]), and the Degasperis–Procesi equation (Eq. (1.1) with $u = v, p = q = 1, a = b = 3$, cf. [3,10]). Apparently, nonlinearity in Eq. (1.2) is quadratic, and thus a natural question rises whether there are integrable CH-type equations with higher-order nonlinearity. Recently, such integrable peakon equations with cubic nonlinearity have been developed: one is the FORQ equation (see [11–14]), and the other one is the Novikov equation (Eq. (1.1) with $u = v, p = q = 2, a = b = 4$, cf, [15,16]).

Cotter, Holm, Ivanov and Percival [17] derived the cross-coupled Camassa–Holm equation (Eq. (1.1) with $a = b = 2$ and $p = q = 1$, which is called the CCCH equation) from a variational principle via an Euler–Lagrange system with the following Lagrangian [17]

$$L(u, v) = \int_{\mathbb{R}} (uv + u_x v_x) dx.$$

Alternatively, it can be formulated as a two-component system of Euler–Poincaré equations in one dimension on $\mathbb{R}$ as follows,

$$\partial_t m = -ad^*_{\delta h/\delta m} m = -(vm)_x - m v_x \quad \text{with} \quad v \doteq \frac{\delta h}{\delta m} = K * n,$$
$$\partial_t n = -ad^*_{\delta h/\delta n} n = -(un)_x - n u_x \quad \text{with} \quad u \doteq \frac{\delta h}{\delta n} = K * m,$$

with $K(x, y) = \frac{1}{2} e^{-|x-y|}$ being the Green function of the Helmholtz operator, and h being the Hamiltonian

$$h(n, m) = \int_{\mathbb{R}} nK * m dx = \int_{\mathbb{R}} mK * n dx.$$

This Hamiltonian system has a two-component singular momentum map [17,18]

$$m(x, t) = \sum_{a=1}^{M} m_a(t)\delta(x - q_a(t)), n(x, t) = \sum_{a=1}^{N} n_b(t)\delta(x - r_b(t)).$$

Such a formal multi-peakon solution, waltzing peakons and compactons of the CCCH are given in [17]. A geometrical interpretation for the CCCH system along with a large class of peakon equations was discussed in [19]. More recently, the Cauchy problem of Eq. (1.1) has been studied extensively. Local well-posedness problem of the CCCH system in Sobolev spaces and Besov spaces is investigated [20,21]. The global existence, blow-up phenomenon, and persistence properties were discussed in [20–22]. The continuity and analyticity were also studied in [23]. It is easy to see that the nonlinearity in CCCH is quadratic, and thus a natural

question rises whether there are CH-type equations with higher-order nonlinearity. In [24], we proposed Eq. (1.1), which models the peculiar wave breaking phenomena. In addition to wave breaking, one of the most interesting aspects of Eq. (1.1) is the existence of waltzing peakons and compactons.

Tracking back to the proof of the well-posedness and non-uniform dependence problem for the CH equation [25–27], the DP equation [28], the b-equation [29–31], the Novikov equation [32], the FORQ equation [33], and the two-component of Camassa–Holm equation [34], we plan to use the Galerkin-type approximation method to deal with Eq. (1.1). In comparison with the CH equation, the Novikov equation, and the two-component CH system, we observe that the well-posedness property for these CH type equations holds for $s > 3/2$ whereas the well-posedness problem for Eq. (1.1) holds for $s > 5/2$. This difference may be caused by the presence of the extra high-order derivative terms $u^{q-1}u_xv_{xx}$ and $v^{p-1}v_xu_{xx}$ in the nonlocal form. These terms make Eq. (1.1) not being treated as a first-order ODE. The novelty in the proof of well-posedness for Eq. (1.1) is the following fact: $\|m\|_{H^{s-2}} = \|u\|_{H^s}$ and $\|n\|_{H^{s-2}} = \|v\|_{H^s}$, which provide the delicate proof for several required nonlinear estimates we need. Let us give the well-posedness of the Cauchy problem for Eq. (1.1) in Sobolev spaces in the sense of Hadamard below.

Theorem 1.1. *Assume that $z_0 = (u_0, v_0) \in H^s \times H^s$ with $s > 5/2$. Let T^* be the maximal existence time of the solution $z = (u, v)$ to Eq. (1.1) with the initial data z_0 for both periodic and non-periodic cases. Then T^* satisfies*

$$T^* \geq T_0 := \frac{2^\kappa - 1}{2^{\kappa+1}\kappa C_s \|z_0\|_{H^s}^\kappa}, \tag{1.3}$$

where C_s is a constant depending on s and $\kappa = \max\{p, q\}$. Moreover, we have

$$\|z(t)\|_{H^s} \leq 2\|z_0\|_{H^s}, \text{ for } 0 \leq t \leq T_0. \tag{1.4}$$

Furthermore, based on the method of approximate solutions and well-posedness estimates, we show that the data-to-solution map is continuous, but not uniformly continuous on any bounded subset of $H^s \times H^s$ with $s > 5/2$. In particular, the novelty of our proof does not employ any conserved quantity, which is a big difference from the proof for the CH equation [25,26] and two-component of Camassa–Holm system [34] relying on the conservation laws in the H^1 norm and also from the proof for the DP equation [28] dependent on the conservation laws in a twisted L^2 norm. Our method is based on the technique of approximate solutions containing terms of both high and low frequencies, which is inspired by the proofs of non-uniform dependence for the Novikov equation [32] and the FORQ equation [33]. We show that the error between the solution of the Cauchy problem (1.1) and the solution of the Cauchy problem with the approximate initial data composed of low and high frequencies may be negligible. Due to the different degrees of nonlinearities ($p \neq q$) in Eq. (1.1), the low and high frequency parts of the approximate solutions $(u^{\omega,\lambda}, v^{\omega,\lambda})$ (see (3.1)–(3.3) below), are more complicated than the usual ones taken in [25,26,28,32–34], but our procedure in this paper may simplify computations.

Theorem 1.2. *If $s > 5/2$, then the data-to-solution map $z_0 = (u_0, v_0) \to z(t) = (u(t), v(t))$ for both the periodic and the non-periodic cases of Eq. (1.1) is not uniformly continuous from any bounded subset of $H^s \times H^s$ into $\mathcal{C}([0, T]; H^s) \times \mathcal{C}([0, T]; H^s)$.*

Theorem 1.2 shows that the data-solution map depends on the initial data continuously, but sharp as established in Theorem 1.1. In other words, Eq. (1.1) is well-posed in Sobolev spaces $H^s \times H^s$ both on the line and the circle for $s > 5/2$, and its data-to-solution map is continuous but not uniformly continuous. Our next result will provide information about stability of the data-solution map. Due to the presence of the high-order derivative terms $u^{q-1}u_xv_{xx}$ and $v^{p-1}v_xu_{xx}$, we need to transform Eq. (1.1) into a system

through differentiating Eq. (1.1) with respect to x (see Eq. (4.2) for details). Then transforming back to the original system leads the data-solution map for Eq. (1.1) to be Hölder continuous in $H^r \times H^r$-topology for all $0 \le r < s$. Let us describe this procedure below.

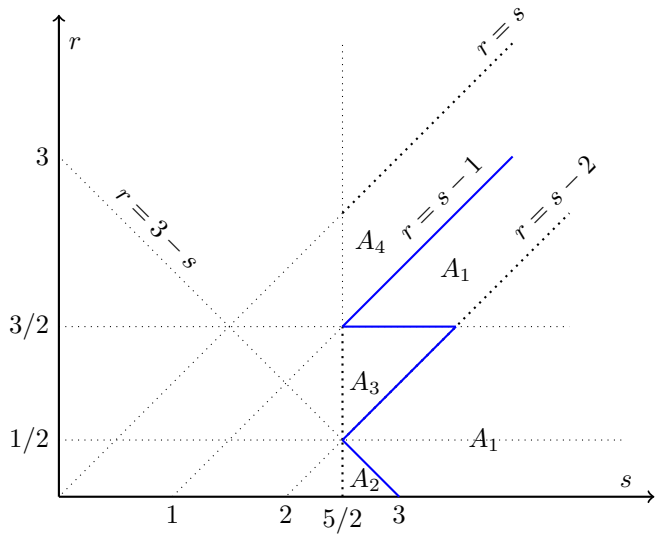
Theorem 1.3. *Let $s > 5/2$ and $0 \le r < s$. Then the data-to-solution map for Eq. (1.1) on the line and the circle is Hölder continuous in $H^s \times H^s$ under the $H^r \times H^r$ norm. In particular, for initial data $z_0 = (u_{0,1}, v_{0,1})$ and $w_0 = (u_{0,2}, v_{0,2})$ in the ball $B(0,\rho) = \{\psi \in H^s \times H^s : \|\psi\|_{H^s} \le \rho\}$ of $H^s \times H^s$, the corresponding solutions $z = (u_1, v_1)$ and $w = (u_2, v_2)$ to Eq. (1.1) satisfy the following inequality*

$$\|z(t) - w(t)\|_{H^r} \le C_{r,s,p,q,\rho} \|z_0 - w_0\|_{H^r}^{\alpha},$$

where the parameter α is given by

$$\alpha = \begin{cases} 1, & (s,r) \in A_1 \doteq \{0 \le r \le 3/2, 3 - r \le r \le s - 2\} \cup \{r > 3/2, r \le s - 1\}, \\ \dfrac{2s-3}{s-r}, & (s,r) \in A_2 \doteq \{(s,r) : 5/2 < s < 3, 0 \le r \le 3 - s\}, \\ \dfrac{s-r}{2}, & (s,r) \in A_3 \doteq \{(s,r) : s > 5/2, s - 2 \le r \le 3/2\}, \\ s - r, & (s,r) \in A_3 \doteq \{(s,r) : s > 5/2, s - 1 \le r < s\}. \end{cases} \tag{1.5}$$

The lifespan T and the constant c only depend on s, r, p, q and ρ.



The entire paper is organized as follows. In Section 2, with the aid of the Galerkin-type approximation scheme we propose, we obtain the well-posedness of the Cauchy problem for Eq. (1.1) in Sobolev spaces in the sense of Hadamard with proving Theorem 1.1. In Section 3, in the light of the Galerkin-type approximation approach and well-posed estimates obtained in Section 2, we show that the data-to-solution map is continuous but not uniformly continuous on any bounded subset of $H^s \times H^s$ with $s > 5/2$ through proving Theorem 1.2. In the last section, we demonstrate that the solution map for Eq. (1.1) is Hölder continuous in H^r-topology for all $0 \le r < s$ through proving Theorem 1.3.

2. Well-posed in the sense of Hadamard

In this section, we shall prove that the Cauchy problem for Eq. (1.1) with the initial data $z_0 = (u_0, v_0) \in H^s \times H^s$ $(s > 5/2)$ is not only well-posed in the sense of Hadamard, but also satisfies the estimate (1.4) on the line and on the circle, namely, prove Theorem 1.1 is true. Let us start from the following lemmas.

Lemma 2.1 (*See [35,36]*). *If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover, we have*
(i) $\|fg\|_{H^r} \le c(\|f\|_{L^\infty}\|g\|_{H^r} + \|g\|_{L^\infty}\|f\|_{H^r})$, *for* $r > 0$.
(ii) $\|fg\|_{H^r} \le c\|f\|_{H^{r+1}}\|g\|_{H^r}$, *for* $r > -1/2$.
(iii) $\|fg\|_{H^{r-1}} \le c\|f\|_{H^{s-1}}\|g\|_{H^{r-1}}$, *for* $0 \le r \le 1, s > 3/2, r + s \ge 2$,
where c is a constant depending only on r, s.

Lemma 2.2 (*See [36,37]*). *If $[D^r, f]g = D^r(fg) - fD^r g$ where $D = (1 - \partial_x^2)^{\frac{1}{2}}$, then we have*
(i) $\|[D^r, f]g\|_{L^2} \le c(\|\partial_x f\|_{L^\infty}\|D^{r-1}g\|_{L^2} + \|D^r f\|_{L^2}\|g\|_{L^\infty}), r > 0$.
(ii) $\|[D^r\partial_x, f]g\|_{L^2} \le c\|f\|_{H^{s-1}}\|g\|_{H^r}, r + 1 \ge 0, s - 1 > 3/2, r + 1 \le s - 1$.
where c is a constant depending only on r.

The proof of Theorem 1.1 consists of the following several steps.

2.1. Priori estimates for u_ϵ and v_ϵ

Applying J_ϵ to system (1.1) leads to the following system

$$\begin{cases} \partial_t J_\epsilon m_\epsilon + J_\epsilon[v_\epsilon^p \partial_x m_\epsilon] + \dfrac{a}{p} J_\epsilon[\partial_x v_\epsilon^p m_\epsilon] = 0, \\ \partial_t J_\epsilon n_\epsilon + J_\epsilon[u_\epsilon^q \partial_x n_\epsilon] + \dfrac{b}{q} J_\epsilon[\partial_x u_\epsilon^q n_\epsilon] = 0, \\ u_{0,\epsilon}(x) = J_\epsilon u_0(x), v_{0,\epsilon}(x) = J_\epsilon v_0(x), \end{cases} \tag{2.1}$$

where the operator J_ϵ is called the Friedrichs mollifier defined by

$$J_\epsilon f(x) = J_\epsilon(f)(x) = j_\epsilon * f, \quad \forall \epsilon \in (0, 1],$$

$j_\epsilon(x) = \frac{1}{\epsilon} j(\frac{x}{\epsilon})$, and $j(x)$ is a $\mathcal{C}^\infty$ function supported in the interval $[-1, 1]$ such that $j(x) \ge 0, \int_{\mathbb{R}} j(x)dx = 1$. Multiplying both sides of the first equation in (2.1) by $D^{s-2}J_\epsilon m_\epsilon D^{s-2}$ and integrating with respect to $x \in \mathbb{R}$, we get

$$\frac{1}{2}\frac{d}{dt}\|J_\epsilon m_\epsilon\|_{H^{s-2}}^2 = \int_{\mathbb{R}} D^{s-2}J_\epsilon m D^{s-2}J_\epsilon(v_\epsilon^p \partial_x m_\epsilon)dx + \frac{a}{p}\int_{\mathbb{R}} D^{s-2}J_\epsilon m_\epsilon D^{s-2}J_\epsilon\left(\partial_x v_\epsilon^p m_\epsilon\right)dx. \tag{2.2}$$

We need to estimate the right-hand side of (2.2). Apparently, both D^s and J_ϵ are commutative and J_ϵ satisfies

$$(J_\epsilon f, g)_0 = (f, J_\epsilon g)_0, \|J_\epsilon u\|_{H^s} \le \|u\|_{H^s}. \tag{2.5}$$

Lemma 2.2(ii) reveals

$$\begin{aligned} &\left|\int_{\mathbb{R}} D^{s-2}J_\epsilon m_\epsilon D^{s-2}J_\epsilon(v_\epsilon^p \partial_x m_\epsilon)dx\right| \lesssim \left|\int_{\mathbb{R}} D^{s-2}J_\epsilon m_\epsilon D^{s-2}J_\epsilon[\partial_x(v_\epsilon^p m_\epsilon) - m_\epsilon \partial_x v_\epsilon^p]dx\right| \\ &\lesssim \left|\int_{\mathbb{R}} [D^{s-2}\partial_x, v_\epsilon^p]J_\epsilon m_\epsilon D^{s-2}J_\epsilon m_\epsilon dx\right| + \left|\int_{\mathbb{R}} v_\epsilon^p D^{s-2}\partial_x J_\epsilon m_\epsilon D^{s-2}J_\epsilon m_\epsilon dx\right| + \|m_\epsilon\|_{H^{s-2}}^2\|v_\epsilon\|_{H^{s-1}}^p \\ &\lesssim \left|\int_{\mathbb{R}} [D^{s-2}\partial_x, v_\epsilon^p]J_\epsilon m_\epsilon D^{s-2}J_\epsilon m_\epsilon dx\right| + \frac{1}{2}\left|\int_{\mathbb{R}} v_\epsilon^p \partial_x(D^{s-2}J_\epsilon m_\epsilon)^2 dx\right| + \|m_\epsilon\|_{H^{s-2}}^2\|v_\epsilon\|_{H^{s-1}}^p \\ &\lesssim \left\|[D^{s-2}\partial_x, v^p]J_\epsilon m\right\|_{L^2}\|m_\epsilon\|_{H^{s-2}} + \frac{1}{2}\left|\int_{\mathbb{R}} \partial_x v^p(D^{s-2}J_\epsilon m)^2 dx\right| + \|m_\epsilon\|_{H^{s-2}}^2\|v_\epsilon\|_{H^{s-1}}^p \\ &\lesssim \|m_\epsilon\|_{H^{s-2}}^2\|v_\epsilon\|_{H^{s-1}}^p. \end{aligned} \tag{2.6}$$

Employing Lemma 2.1(ii) and (2.5) yields

$$\left|\int_{\mathbb{R}} D^{s-2}J_\epsilon m_\epsilon D^{s-2}J_\epsilon\left(\partial_x v_\epsilon^p m_\epsilon\right)dx\right| \lesssim \|v_\epsilon\|_{s-1}^p \|m_\epsilon\|_{H^{s-2}}^2. \tag{2.7}$$

For all $s \in \mathbb{R}$, we have

$$\|u_\epsilon\|_{H^s} = \|m_\epsilon\|_{H^{s-2}} \text{ and } \|v_\epsilon\|_{H^s} = \|n_\epsilon\|_{H^{s-2}}. \tag{2.8}$$

Therefore,

$$\frac{d}{dt}\|u_\epsilon(t)\|_{H^s} \lesssim C_s\left(\|u_\epsilon\|_{H^s} + \|v_\epsilon\|_{H^s}\right)^{p+1}.$$

Adopting a similar procedure for v_ϵ produces

$$\frac{d}{dt}\|z_\epsilon\|_{H^s} \doteq \frac{d}{dt}\left(\|u_\epsilon\|_{H^s} + \|v_\epsilon\|_{H^s}\right) \lesssim 2C_s\left(\|u_\epsilon\|_{H^s} + \|v_\epsilon\|_{H^s}\right)^{\kappa+1} = 2C_s\|z_\epsilon\|_{H^s}^{\kappa+1}, \tag{2.9}$$

where $\kappa = \max\{p, q\}$.

Solving the differential inequality (2.9) generates

$$\|z_\epsilon(t)\|_{H^s} \le \frac{\|z_0\|_{H^{s-1}}}{\sqrt[\kappa]{1 - 2\kappa C_s\|z_0\|_{H^s}^\kappa t}}. \tag{2.10}$$

Let $T_0 = \frac{2^\kappa - 1}{2^{\kappa+1}\kappa C_s\|z_0\|_{H^s}^\kappa}$, then from Eq. (2.10) we see that there exist the solutions u, v for $0 \le t \le T_0$ with the following bound

$$\|z(t)\|_{H^s} \le 2\|z_0\|_{H^s}, \text{ for } 0 \le t \le T_0. \tag{2.11}$$

Moreover, by Eq. (2.1) we may obtain the following estimates for $\partial_t u_\epsilon(t)$ and $\partial_t v_\epsilon(t)$:

$$\begin{aligned}
&\|\partial_t u_\epsilon(t)\|_{H^{s-1}} \cong \|\partial_t m_\epsilon(t)\|_{H^{s-3}} \lesssim \|v^p\partial_x J_\epsilon m\|_{H^{s-3}} + \|\partial_x(J_\epsilon v)^p J_\epsilon m\|_{H^{s-3}} \lesssim \|z_0\|_{H^s}^{p+1},\\
&\|\partial_t v_\epsilon(t)\|_{H^{s-1}} \cong \|\partial_t n_\epsilon(t)\|_{H^{s-3}} \lesssim \|z_0\|_{H^s}^{q+1}.
\end{aligned} \tag{2.12}$$

2.2. Existence of solutions on the line

Theorem 2.1. *There exists a solution $z = (u, v)$ to the Cauchy problem (2.1) in the space $\mathcal{C}([0, T]; H^s \times H^s)$ with $s > 5/2$. Furthermore, the $H^s \times H^s$ norm of z satisfies Eqs.(2.11)* and *(2.12).*

So far, we have studied the existence of a unique solution $z_\epsilon \in \mathcal{C}([0, T]; H^s \times H^s), s > 5/2$ to the initial value problem (2.1) with life span $T = \frac{2^\kappa - 1}{2^{\kappa+1}\kappa C_s\|z_0\|_{H^s\times H^s}^\kappa}$ as well as the size estimates (2.11) and (2.12). Next, we need to show that $z_\epsilon \to z \doteq (u, v) \in \mathcal{C}([0, T]; H^s \times H^s)$ where z is the solution to Eq. (1.1). Our proof is carried out through refining the convergence of the family $\{z_\epsilon\} = \{u_\epsilon, v_\epsilon\}$ several times by extracting its subsequences. After each extraction, for our convenient discussion and simplicity, the resulting subsequence is still labeled as $\{z_\epsilon\} = \{u_\epsilon, v_\epsilon\}$.

Weak convergence in $L^\infty(I; H^s \times H^s)$. The set of functions $\{z_\epsilon\}_{\epsilon\in(0,1]}$ is bounded in the space $\mathcal{C}(I; H^s \times H^s) \subset L^\infty(I; H^s \times H^s)$. By the inequality (2.11), we have

$$\begin{aligned}
&\|z_\epsilon\|_{L^\infty(I;H^s\times H^s)} = \sup_{t\in I}\|z_\epsilon\|_{H^s\times H^s} \le 4\|z_0\|_{H^s\times H^s},\\
\Rightarrow &\{z_\epsilon\}_{\epsilon\in(0,1]} \subset \overline{B}(0, 2\|z_0\|_{H^s\times H^s}) \subset L^\infty(I; H^s \times H^s),
\end{aligned} \tag{2.13}$$

Alaoglu's theorem tells us that $\{z_\epsilon\}$ is pre-compact in $\overline{B}(0, 2\|z_0\|_{H^s\times H^s}) \subset L^\infty(I; H^s \times H^s)$ with respect to the weak topology. Therefore, we may extract a subsequence $\{z_{\epsilon'}\}$ that converges to an element $z \in$

$\overline{B}(0, 2\|z_0\|_{H^s\times H^s})$ weakly* sense. Provided we have such a construction, then z would be our desired function satisfying the solution estimate (1.4).

Convergence in $\mathcal{C}^\sigma(I; H^{s-\sigma} \times H^{s-\sigma})$ with $\sigma \in (0,1)$. We shall apply the Ascoli's theorem to prove convergence in the spaces. Let $\phi \in \mathcal{S}(\mathbb{R})$ and $\phi(x) > 0, \forall x \in \mathbb{R}$. We shall prove that there exists a subsequence of $\{\phi z_\epsilon(t)\}$ converging to a function $\phi z(t) \in \mathcal{C}(I; H^{s-\sigma} \times H^{s-\sigma})$. Define $z(t) \doteq \{\phi z_\epsilon(t)\}_{\epsilon\in(0,1]}$. Then we know that for each $t \in I$ the set $z(t) \subset H^s \times H^s$ is pre-compact in $H^{s-\sigma} \times H^{s-\sigma}$ as a consequence of the Rellich's theorem. Thus, the first condition of Ascoli's theorem is met. It remains to show that the second condition is also met, i.e. that $\phi z_\epsilon(t)$ is equicontinuous. To see this, we shall first show $z_\epsilon \in \mathcal{C}^\sigma(I, H^{s-\sigma} \times H^{s-\sigma})$ for $\epsilon, \sigma \in (0,1)$., Then, we prove that the $\mathcal{C}^\sigma(I, H^{s-\sigma} \times H^{s-\sigma})$ norm of z_ϵ satisfies

$$\|z_\epsilon(t)\|_{\mathcal{C}^\sigma(I,H^{s-\sigma}\times H^{s-\sigma})} \lesssim \|z_0\|_{H^s\times H^s} + \|z_0\|^{\kappa+1}_{H^s\times H^s}. \tag{2.14}$$

Let us begin with the following norm definition

$$\|z_\epsilon(t)\|_{\mathcal{C}^\sigma(I,H^{s-\sigma}\times H^{s-\sigma})} \doteq \sup_{t\in I} \|z_\epsilon(t)\|_{H^{s-\sigma}\times H^{s-\sigma}} + \sup_{t_1\neq t_2} \frac{\|z_\epsilon(t_1) - z_\epsilon(t_2)\|_{H^{s-\sigma}\times H^{s-\sigma}}}{|t_1 - t_2|^\sigma}. \tag{2.15}$$

Because the first term on the right hand side of (2.15) is bounded by Eq. (2.11), we have

$$\sup_{t\in I} \|z_\epsilon(t)\|_{H^{s-\sigma}\times H^{s-\sigma}} \leq \sup_{t\in I} \|z_\epsilon(t)\|_{H^s\times H^s} \leq 2\|z_0\|_{H^s\times H^s}.$$

The second term on the right hand side of (2.15), due to the inequality $x^\sigma \leq 1 + x$, generates the following estimate:

$$\begin{aligned}
\sup_{t_1\neq t_2} \frac{\|z_\epsilon(t_1) - z_\epsilon(t_2)\|_{H^{s-\sigma}\times H^{s-\sigma}}}{|t_1 - t_2|^\sigma} &= \sup_{t_1\neq t_2} \left(\int_{\mathbb{R}} (1+\xi^2)^s \frac{|\hat{z}_\epsilon(\xi, t_1) - \hat{z}_\epsilon(\xi, t_2)|^2}{(1+\xi^2)^\sigma |t_1 - t_2|^{2\sigma}} d\xi \right)^{1/2} \\
&\leq \sup_{t_1\neq t_2} \left(\int_{\mathbb{R}} (1+\xi^2)^s \left[1 + \frac{1}{(1+\xi^2)|t_1 - t_2|^2} \right] |\hat{z}_\epsilon(\xi, t_1) - \hat{z}_\epsilon(\xi, t_2)|^2 d\xi \right)^{1/2} \\
&\leq \sup_{t\in I} (2\|z_\epsilon(t)\|_{H^s\times H^s} + \|\partial_t z_\epsilon(t)\|_{H^{s-1}\times H^{s-1}}) \\
&\lesssim \|z_0\|_{H^s\times H^s} + \|z_0\|^{\kappa+1}_{H^s\times H^s},
\end{aligned}$$

where (2.11) and (2.12) are applied in the last inequality. Combining these bounds together leads to the desired estimate (2.14). On the other hand, the inequality (2.14) apparently implies

$$\begin{aligned}
\|z_\epsilon(t_1) - z_\epsilon(t_2)\|_{H^{s-\sigma}\times H^{s-\sigma}} &\leq \|z_\epsilon(t)\|_{\mathcal{C}^\sigma(I,H^{s-\sigma}\times H^{s-\sigma})} |t_1 - t_2|^\sigma \\
&\lesssim \left(\|z_0\|_{H^s\times H^s} + \|z_0\|^{\kappa+1}_{H^s\times H^s} \right) |t_1 - t_2|^\sigma, \quad t_1, t_2 \in I, \quad \forall \epsilon, \sigma \in (0,1).
\end{aligned} \tag{2.16}$$

Furthermore, we have the following equicontinuity property for $\{\phi z_\epsilon(t)\}_{\epsilon\in(0,1]}$

$$\begin{aligned}
\|\phi z_\epsilon(t_1) - \phi z_\epsilon(t_2)\|_{H^{s-\sigma}\times H^{s-\sigma}} &\leq \|\phi\|_{H^{s-\sigma}} \|z_\epsilon(t_1) - z_\epsilon(t_2)\|_{H^{s-\sigma}\times H^{s-\sigma}} \\
&\lesssim \left(\|z_0\|_{H^s\times H^s} + \|z_0\|^{\kappa+1}_{H^s\times H^s} \right) |t_1 - t_2|^\sigma.
\end{aligned} \tag{2.17}$$

The Ascoli's theorem admits

$$\|\phi z_\epsilon - \phi z\|_{\mathcal{C}^\sigma(I,H^{s-\sigma}\times H^{s-\sigma})} = \sup_{t\in I} \|\phi z_\epsilon(t) - \phi z(t)\|_{H^{s-\sigma}\times H^{s-\sigma}} \to 0 \text{ for } \epsilon \to 0. \tag{2.18}$$

Convergence in $\mathcal{C}(I; \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R}))$. Let σ satisfy $s - 2 - \sigma > 3/2$. Then, applying the Sobolev lemma and (2.18) reveals

$$\begin{aligned}
\|\phi z_\epsilon - \phi z\|_{\mathcal{C}(I,\mathcal{C}^1(\mathbb{R})\times\mathcal{C}^1(\mathbb{R}))} &= \sup_{t\in I} \|\phi z_\epsilon(t) - \phi z(t)\|_{\mathcal{C}^1(\mathbb{R})\times\mathcal{C}^1(\mathbb{R})} \\
&\lesssim \|\phi z_\epsilon(t) - \phi z(t)\|_{H^{s-\sigma}\times H^{s-\sigma}} \to 0 \text{ for } \epsilon \to 0.
\end{aligned} \tag{2.19}$$

Therefore, convergence in $\mathcal{C}(I; \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R}))$ has been established. Next, we prove that z solves Eq. (1.1).

Verification of z being a solution to the Cauchy problem (1.1). Let us first recall the generalization of Sobolev spaces from the real analysis. Suppose that a system of functions $f_n : I \to \mathbb{R}$ are continuous. If there is some $t_0 \in I$ such that $f_n(t_0) \to f(t_0)$ as $n \to \infty$ and f_n' is uniformly convergent to $f'(t)$ on I, then f_n also uniformly converges to f on I and $f'(t) = \lim_{n\to\infty} f_n'(t)$. We shall apply this result to the sequence $\{z_\epsilon(t)\}_{\epsilon\in(0,1]}$. we already show $\{z_\epsilon(t)\}_{\epsilon\in(0,1]}$ is convergent to $z(t)$ in $\mathcal{C}(I;\mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R}))$, which implies the first condition of the theorem is satisfied. From the convergence $\phi z_\epsilon \to \phi z$ in $\mathcal{C}(I;\mathcal{C}^1(\mathbb{R})\times\mathcal{C}^1(\mathbb{R}))$, we know that $z_\epsilon \to z$ and $\partial_x z_\epsilon \to \partial_x z$ are pointwisely convergent. This reveals that the right hand side of (2.1) converges to the corresponding terms without ϵ. The remaining task is to show that $\partial_t(\phi z_\epsilon) = (\partial_t(\phi u_\epsilon), \partial_t(\phi v_\epsilon))$ in $\mathcal{C}(I;\mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R}))$ is uniformly convergent.

Due to m and n are in a paralleled situation, let us only consider the component m while the other component can be treated in the same way. Starting from the first equation of (2.1) and multiplying both sides of the equation by ϕ, we have

$$\partial_t(\phi m_\epsilon) = -\phi J_\epsilon(v_\epsilon^p \partial_x m_\epsilon) - \frac{a}{p}\phi J_\epsilon(\partial_x v_\epsilon^p m_\epsilon). \tag{2.20}$$

Casting σ into $s - \sigma - 3 > -1/2$ with $s - \sigma - 3 \neq 1$ allows us to derive the equicontinuity. Regarding the first term on the right-hand-side of (2.20), Lemma 2.1(ii) tells us

$$\begin{aligned}
&\| \phi J_\epsilon[v_\epsilon^p(t_1)\partial_x m_\epsilon(t_1)] - \phi J_\epsilon[v_\epsilon^p(t_2)\partial_x m_\epsilon(t_2)] \|_{H^{s-\sigma-3}} \\
&\qquad \lesssim \|v_\epsilon^p(t_1)\|_{H^{s-\sigma-2}} \|m_\epsilon(t_1) - m_\epsilon(t_2)\|_{H^{s-\sigma-2}} + \|m_\epsilon(t_1)\|_{H^{s-\sigma-2}} \|v_\epsilon^p(t_1) - v_\epsilon(t_2)^p\|_{H^{s-\sigma-2}} \\
&\qquad \lesssim \|v_\epsilon^p(t_1)\|_{H^{s-\sigma-2}} \|m_\epsilon(t_1) - m_\epsilon(t_2)\|_{H^{s-\sigma-2}} \\
&\qquad\quad + \|m_\epsilon(t_1)\|_{H^{s-\sigma-2}} \|v_\epsilon(t_1) - v_\epsilon(t_2)\|_{H^{s-\sigma-2}} \sum_{j=0}^{p-1} \|v_\epsilon(t_1)\|_{H^{s-\sigma-2}}^{p-1-j} \|v_\epsilon(t_2)\|_{H^{s-\sigma-2}}^{j},
\end{aligned}$$

which yields

$$\| \phi J_\epsilon[v_\epsilon^p(t_1)\partial_x m_\epsilon(t_1)] - \phi J_\epsilon[v_\epsilon^p(t_2)\partial_x m_\epsilon(t_2)] \|_{H^{s-\sigma-3}} \lesssim \left(\|z_0\|_{H^s\times H^s}^{p+1} + \|z_0\|_{H^s\times H^s}^{p+\kappa+1} \right) |t_1 - t_2|^\sigma.$$

The second term on the right hand side of Eq. (2.20) can similarly be estimated

$$\|\phi J_\epsilon[\partial_x v_\epsilon^p(t_1) m_\epsilon(t_1) - \partial_x v_\epsilon^p(t_2) m_\epsilon(t_2)]\|_{H^{s-\sigma-3}} \lesssim \left(\|z_0\|_{H^s\times H^s}^{p+1} + \|z_0\|_{H^s\times H^s}^{p+\kappa+1} \right) |t_1 - t_2|^\sigma.$$

Therefore, by Ascoli's theorem, we conclude that a subsequence of $\{\phi z_\epsilon(t)\}_{\epsilon\in(0,1]}$ satisfies

$$\partial_t(\phi Z_\epsilon) = \begin{cases} \partial_t(\phi m_\epsilon) \to -\phi v^p m_x - \dfrac{a}{p}\phi v_x^p m, \\ \partial_t(\phi n_\epsilon) \to -\phi u^q n_x - \dfrac{b}{q}\phi u_x^q n, \end{cases} \quad \text{in } \mathcal{C}(I; H^{s-\sigma-3}). \tag{2.21}$$

As per $s - \sigma - 1 > 1/2$ and the Sobolev lemma, we find that $\mathcal{C}(I, \mathcal{C}(\mathbb{R}) \times \mathcal{C}(\mathbb{R})) \hookrightarrow \mathcal{C}(I; H^{s-\sigma-1} \times H^{s-\sigma-1})$.

Claim: $z \in L^\infty(I; H^s \times H^s) \cap Lip(I; H^{s-1} \times H^{s-1})$ **is a solution to** (1.1). First, we notice that $\phi z_\epsilon \to \phi z$ and $\partial_t(\phi z_\epsilon) \to \partial_t(\phi z)$ in the space $\mathcal{C}(I, \mathcal{C}(\mathbb{R}) \times \mathcal{C}(\mathbb{R}))$, which imply that $t \mapsto \phi z(t)$ is a differentiable map. As we chose ϕ with no zeros, the formula $\partial_t(\phi z_\epsilon) = \phi \partial_t(z_\epsilon)$ allows us to get rid of ϕ and ϵ in (2.21). Thus, we are able to locate $z \in Lip(I; H^s \times H^s)$ with the following property

$$\|z(t_1) - z(t_2)\|_{H^s\times H^s} \le \sup_{t\in I} \|\partial_t z(t)\|_{H^s\times H^s} |t_1 - t_2| \lesssim \|z_0\|_{H^s\times H^s}^{\kappa+1} |t_1 - t_2|. \tag{2.22}$$

Regularity improvement of z up to $\mathcal{C}(I; H^s \times H^s)$. We already know $z \in L^\infty(I; H^s \times H^s) \cap Lip(I; H^{s-1} \times H^{s-1})$. Let us now prove $z \in \mathcal{C}(I; H^s \times H^s)$, namely, if $t_n \in I$ converge to $t \in I$ as n goes to infinity, then $\lim_{n\to\infty} \|z(t_n) - z(t)\|_{H^s\times H^s} = 0$. According to the norm definition in $H^s \times H^s$, this is equivalent to showing

$$\lim_{n\to\infty} \left(\|z(t_n)\|_{H^s\times H^s}^2 - \langle z(t_n), z(t)\rangle_{H^s\times H^s} + \|z(t)\|_{H^s\times H^s}^2 \right) = 0. \tag{2.23}$$

Lemma 2.3. *The solution $z \in L^\infty(I; H^s \times H^s) \cap Lip(I; H^{s-1} \times H^{s-1})$ is continuous on I in the sense of weak topology in $H^s \times H^s$, i.e.*

$$\langle z(t_n) - z(t), \varphi\rangle_{H^s\times H^s} = 0, \text{ for any } \varphi \in H^s \times H^s.$$

Proof. Let $\varphi \in H^s \times H^s$. For any $\epsilon > 0$, choose $\psi \in \mathcal{S}(\mathbb{R})$ such that $\|\varphi - \psi\|_{H^s\times H^s} < \epsilon/(4\|z_0\|_{H^s\times H^s})$. The triangle inequality yields

$$\begin{aligned}|\langle z(t_n) - z(t), \varphi\rangle_{H^s\times H^s}| &\le |\langle z(t_n) - z(t), \varphi - \psi\rangle_{H^s\times H^s}| + |\langle z(t_n) - z(t), \psi\rangle_{H^s\times H^s}| \\ &\le \|z(t_n) - z(t)\|_{H^{s-2}}\|\varphi - \psi\|_{H^s\times H^s} + \|z(t_n) - z(t)\|_{H^s\times H^s}\|\psi\|_{H^s\times H^s} \\ &\le \epsilon/2 + \|z_0\|^{\kappa+1}_{H^s\times H^s}\|\psi\|_{H^s}|t_n - t|,\end{aligned}$$

where the inequalities (2.21) and (2.11) are applied. Since $\|\psi\|_{H^s}$ is bounded, we select N such that for any $n > N$, $|t_n - t| < \epsilon/(2\|z_0\|^{\kappa+1}_{H^s\times H^s}\|\psi\|_{H^s})$. Hence, we have

$$|\langle z(t_n) - z(t), \varphi\rangle_{H^s\times H^s}| < \epsilon, \forall n > N,$$

which concludes the proof. □

Employing Lemma 2.3 reduces (2.23) to $\lim_{n\to\infty} \|z(t_n)\|_{H^s\times H^s} = \|z(t)\|_{H^s\times H^s}$, i.e. we prove the map $t \mapsto \|z(t)\|_{H^s\times H^s}$ is continuous. We already know that $\|J_\epsilon z(t)\|_{H^s\times H^s}$ converges to $\|z(t)\|_{H^s\times H^s}$ pointwise in t as $\epsilon \to 0$. Thus, it suffices to show that each $\|J_\epsilon z(t)\|_{H^s\times H^s}$ is Lipschitz with the bounded Lipschitz constants for the whole family of functions. Taking a similar work procedure of $\|z_\epsilon\|_{H^s\times H^s}$ in Section 2.1 we arrive at $\frac{d}{dt}\|J_\epsilon z(t)\|_{H^s\times H^s} \lesssim \|z_0\|^{\kappa+1}_{H^s\times H^s}$. Therefore, we conclude that $\|z(t)\|_{H^s\times H^s}$ is Lipschitz and the solution z is in $\mathcal{C}(I; H^s \times H^s)$.

2.3. Uniqueness of solution on a line

We have already shown that there exists a solution $Z = (m, n)$ to the Cauchy problem (1.1) in $\mathcal{C}(I; H^{s-2})$, which satisfies the estimates (2.11) and (2.12) with lifespan $T = \frac{2^\kappa - 1}{2^{\kappa+1}\kappa C_s\|z_0\|^\kappa_{H^s}}$. In this section we shall prove the solution is unique.

Theorem 2.2. *For the initial data $z_0 \in H^s \times H^s$ with $s > 5/2$, the Cauchy problem (1.1) has a unique solution $z = (u, v)$ in the space $\mathcal{C}(I; H^s)$.*

Proof. Let $z_0 \in H^s \times H^s$ and $s > 5/2$. Suppose $Z_1 \doteq (u_1, v_1, m_1, n_1)$ and $Z_2 \doteq (u_2, v_2, m_2, n_2)$ are two solutions to the Cauchy problem (1.1) with $u_1(x, 0) = u_0(x) = u_2(x, 0)$ and $v_1(x, 0) = v_0(x) = v_2(x, 0)$. Let $M = m_1 - m_2, N = n_1 - n_2, U = u_1 - u_2, V = V_1 - V_2$, then we have

$$\begin{cases} \partial_t M + \partial_x(v_1^p M) = -\partial_x[(v_1^p - v_2^p)m_2] + \dfrac{p-a}{p}\partial_x v_1^p M + \dfrac{p-a}{p}\partial_x(v_1^p - v_2^p)m_2, \\ \partial_t N + \partial_x(u_1^q N) = -\partial_x[(u_1^q - u_2^q)n_2] + \dfrac{q-b}{q}\partial_x u_1^q N + \dfrac{q-b}{q}\partial_x(u_1^q - u_2^q)n_2, \\ M|_{t=0} = N|_{t=0} = 0. \end{cases}$$

Without losing generality, let us just consider the third equation while other two can be handled in a similar way. Assume $\sigma \in (1/2, s-3)$, then applying D^σ to the first equation and multiplying the result by $D^\sigma E$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|M\|^2_{H^\sigma} = &-\int_{\mathbb{R}} D^\sigma M D^\sigma \partial_x(v_1^p M)dx \\ &-\int_{\mathbb{R}} D^\sigma M D^\sigma\left(-\partial_x[(v_1^p - v_2^p)m_2] + \frac{p-a}{p}\partial_x v_1^p M + \frac{p-a}{p}\partial_x(v_1^p - v_2^p)m_2\right)dx.\end{aligned} \tag{2.24}$$

Apparently, one can verify the following inequalities

$$\begin{aligned}\left|\int_{\mathbb{R}} D^\sigma M D^\sigma \partial_x (v_1^p M) dx\right| &\lesssim \left|\int_{\mathbb{R}} D^\sigma M \left([D^\sigma \partial_x, v_1^p] M + v_1^p \partial_x D^\sigma M\right) dx\right| \\ &\lesssim \|[D^\sigma \partial_x, v_1^p] M\|_{L^2} \|M\|_{H^\sigma} + \frac{1}{2}\left|\int_{\mathbb{R}} \partial_x v_1^p (D^\sigma M)^2 dx\right| \\ &\lesssim \|v_1^p\|_{H^\rho} \|M\|_{H^\sigma}^2 + \|(v_1^p)_x\|_{L^\infty} \|M\|_{H^\sigma}^2 \\ &\lesssim \|v_1^p\|_{H^s} \|M\|_{H^\sigma}^2,\end{aligned} \tag{2.25}$$

where Lemma 2.2(ii) with $\rho > 3/2$ and $\sigma + 1 \le \rho$ are applied.

Since Z' and Z satisfy the same estimate (2.11), for the first term on the right hand side of (2.24) we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}} D^\sigma M D^\sigma \partial_x \left[(v_1^p - v_2^p) m_2\right] dx\right| \lesssim \|M\|_{H^\sigma} \left|\int_{\mathbb{R}} \left([D^\sigma \partial_x, v_1^p] m_2 + (v_1^p - v_2^p) \partial_x D^\sigma m_2\right) dx\right| \\ &\lesssim \|[D^\sigma \partial_x, (v_1^p - v_2^p)] m_2\|_{L^2} \|M\|_{H^\sigma} + \frac{1}{2}\left|\int_{\mathbb{R}} \partial_x (v_1^p - v_2^p) D^\sigma m_2 dx\right| \|M\|_{H^\sigma} \\ &\lesssim \|v_1^p - v_2^p\|_{H^\rho} \|m_2\|_{H^\sigma} \|M\|_{H^\sigma} + \|(v_1^p - v_2^p)_x\|_{L^\infty} \|m_2\|_{H^\sigma} \|M\|_{H^\sigma} \\ &\lesssim \|M\|_{H^\sigma} \|N\|_{H^\sigma} \|u_2\|_{H^s} \sum_{i=0}^{p-1} \|v_1\|_{H^s}^{p-1-i} \|v_2\|_{H^s}^i.\end{aligned} \tag{2.26}$$

Meanwhile, the nonlocal term on the right-hand side of (2.24) becomes

$$\begin{aligned}&\left|\int_{\mathbb{R}} D^\sigma M D^\sigma \left(\frac{p-a}{p} \partial_x v_1^p M + \frac{p-a}{p} \partial_x (v_1^p - v_2^p) m_2\right) dx\right| \\ &\lesssim \|M\|_{H^\sigma} \|\partial_x v_1^p M\|_{H^\sigma} + \|M\|_{H^\sigma} \|\partial_x (v_1^p - v_2^p) m_2\|_{H^\sigma} \\ &\lesssim \|v_1^p\|_{H^s} \|M\|_{H^\sigma}^2 + \|M\|_{H^\sigma} \|N\|_{H^\sigma} \|u_2\|_{H^s} \sum_{i=0}^{p-1} \|v_1\|_{H^s}^{p-1-i} \|v_2\|_{H^s}^i,\end{aligned} \tag{2.27}$$

where Lemma 2.1(ii) with $\sigma \in (1/2, s-3)$ is used.

Combining (2.26) with (2.27) gives

$$\frac{1}{2}\frac{d}{dt}\|M\|_{H^\sigma}^2 \lesssim \|z_0\|_{H^s \times H^s}^p \left(\|M\|_{H^\sigma}^2 + \|M\|_{H^\sigma} \|N\|_{H^\sigma}\right), \ \forall \sigma \in (1/2, s-2),$$

which yields

$$\frac{d}{dt}\|M\|_{H^\sigma} \lesssim \|z_0\|_{H^s \times H^s}^{p/2} \left(\|M\|_{H^\sigma} + \|N\|_{H^\sigma}\right).$$

Considering a similar estimate by $\frac{d}{dt}\|N\|_{H^\sigma}$, we obtain

$$\frac{d}{dt}\left(\|M\|_{H^\sigma} + \|N\|_{H^\sigma}\right) \lesssim \left(\|z_0\|_{H^s \times H^s}^{p/2} + \|z_0\|_{H^s \times H^s}^{q/2}\right)\left(\|M\|_{H^\sigma} + \|N\|_{H^\sigma}\right).$$

Solving this inequality yields

$$\|M\|_{H^\sigma} + \|N\|_{H^\sigma} \lesssim \left(\|M(0)\|_{H^\sigma} + \|N(0)\|_{H^\sigma}\right) \exp(\|z_0\|_{H^s \times H^s}^{p/2} + \|z_0\|_{H^s \times H^s}^{q/2}) t,$$

which implies $M = N = 0$ due to $M(0) = N(0) = 0$. Furthermore, we have $u_1 - u_2 = U = \frac{1}{2} e^{-|x|} * M = 0$, $v_1 - v_2 = U = \frac{1}{2} e^{-|x|} * N = 0$ which convey the solution to (1.1) is unique in the spaces $\mathcal{C}(I; H^s)$. So, we obtain the size estimate and lifespan for the solution (u, v) given in Theorem 1.1. □

2.4. Continuity of the data-to-solution map on a line

In this section, we shall complete the proof of the Hadamard well-posedness for the Cauchy problem (1.1) on the line through showing that the data-to-solution map $z_0 = (u_0(x), v_0(x)) \mapsto z(x,t) = (u(x,t), v(x,t)) \in \mathcal{C}(I; H^s) \times \mathcal{C}(I; H^s)$ is continuous. More precisely speaking,

Theorem 2.3. *Assume $z_k(x,t) = (u_k(x,t), v_k(x,t))$ and $z(x,t) = ((u(x,t), v(x,t)))$ are the solutions corresponding to the initial data $z_{0,k}(x) = (u_{0,k}(x), v_{0,k}(x))$ and $z_0(x) = (u_0(x), v_0(x))$ respectively, and $z_{0,k}(x) = (u_{0,k}(x), v_{0,k}(x)) \to (u_0(x), v_0(x))$ in $H^s \times H^s$, then we have $z_{0,k}(x) = (u_k(x,t), v_k(x,t)) \to z(x,t) = ((u(x,t), v(x,t)))$ in $\mathcal{C}(I; H^s) \times \mathcal{C}(I; H^s)$.*

Due to the presence of the high-order derivative terms $u_x v_{xx}$ and $v_x u_{xx}$, we employ the approach of transforming the original solution $z(x,t) = (u(x,t), v(x,t))$ into the solution $Z(x,t) = (u(x,t), v(x,t), m(x,t), n(x,t))$ given by Eq. (1.1), and use the convolution operator J_ϵ ($\epsilon \in (0,1]$) to smooth out the initial data. Let $Z^\epsilon = (u^\epsilon, v^\epsilon, m^\epsilon, n^\epsilon)$ be the solution to Eq. (1.1) with initial data $J_\epsilon Z_0 = (j_\epsilon * u_0, j_\epsilon * v_0, j_\epsilon * (u_0 - \partial_x^2 u_0), j_\epsilon * (u_0 - \partial_x^2 u_0))$ and $Z_k^\epsilon = (u_k^\epsilon, v_k^\epsilon, m_k^\epsilon, n_k^\epsilon)$ be the solution with initial data $J_\epsilon Z_{0,k} = (j_\epsilon * u_{0,k}, j_\epsilon * v_{0,k}, j_\epsilon * (u_{0,k} - \partial_x^2 u_{0,k}), j_\epsilon * (u_{0,k} - \partial_x^2 u_{0,k}))$.

By the following triangle inequality

$$\|z_k - z\|_{\mathcal{C}(I;H^s)} \le \|z_k - z_k^\epsilon\|_{\mathcal{C}(I;H^s)} + \|z_k^\epsilon - z^\epsilon\|_{\mathcal{C}(I;H^s)} + \|z^\epsilon - z\|_{\mathcal{C}(I;H^s)}, \tag{2.28}$$

we will prove that, for any $n > N$, each of these terms can be bounded by $\eta/3$ for suitable choices of ϵ and N, but ϵ only depends on η, whereas the choice of N is dependent of both η and ϵ.

Estimation of $\|z_k^\epsilon - z^\epsilon\|_{\mathcal{C}(I;H^s)}$, $\|z_k^\epsilon - z^\epsilon\|_{\mathcal{C}(I;H^s)}$ and $\|z^\epsilon - z\|_{\mathcal{C}(I;H^s)}$. Let $U^\epsilon = u^\epsilon - u_k^\epsilon$, $V^\epsilon = v^\epsilon - v_k^\epsilon$, $M^\epsilon = m^\epsilon - m_k^\epsilon$, $N^\epsilon = n^\epsilon - n_k^\epsilon$. Then through a direct calculation we know that $(U^\epsilon, V^\epsilon, M^\epsilon, N^\epsilon)$ solves the following equation

$$\begin{cases} \partial_t M^\epsilon + \partial_x[(v^\epsilon)^p M^\epsilon] = -\partial_x\{[(v^\epsilon)^p - (v_k^\epsilon)^p]m_k^\epsilon\} + \dfrac{p-a}{p}\partial_x(v^\epsilon)^p M^\epsilon + \dfrac{p-a}{p}\partial_x[(v^\epsilon)^p - (v_k^\epsilon)^p]m_k^\epsilon, \\ \partial_t N^\epsilon + \partial_x[(u^\epsilon)^q N^\epsilon] = -\partial_x\{[(u^\epsilon)^q - (u_k^\epsilon)^q]n_k^\epsilon\} + \dfrac{q-b}{q}\partial_x(u^\epsilon)^q N^\epsilon + \dfrac{q-b}{q}\partial_x[(u^\epsilon)^q - (u_k^\epsilon)^q]n_k^\epsilon, \\ M = U - \partial_x^2 U, N = V - \partial_x^2 V, \quad U|_{t=0} = J_\epsilon u_0 - J_\epsilon u_{0,k}, V|_{t=0} = J_\epsilon v_0 - J_\epsilon v_{0,k}. \end{cases}$$

For the sake of simplicity, let us only consider the third equation while other two cases can be treated in a similar way. Applying the operator D^{s-2} to the left hand side of the first equation, multiplying by $D^{s-2}P$, and then integrating over $\mathbb{R}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|M^\epsilon\|_{H^{s-2}}^2 &= \int_{\mathbb{R}} D^{s-2}M^\epsilon D^{s-2}\partial_x[(v^\epsilon)^p M^\epsilon]dx \\ &+ \int_{\mathbb{R}} D^{s-2}M^\epsilon D^{s-2}\left(-\partial_x\{[(v^\epsilon)^p - (v_k^\epsilon)^p]m_k^\epsilon\} + \frac{p-a}{p}\partial_x(v^\epsilon)^p M^\epsilon + \frac{p-a}{p}\partial_x[(v^\epsilon)^p - (v_k^\epsilon)^p]m_k^\epsilon\right)dx. \end{aligned} \tag{2.29}$$

Apparently, we have the following fact

$$\begin{aligned} \left|\int_{\mathbb{R}} D^{s-2}M^\epsilon D^{s-2}\partial_x[(v^\epsilon)^p M^\epsilon]dx\right| &\lesssim \left|\int_{\mathbb{R}} [D^{s-2}\partial_x, (v^\epsilon)^p]M^\epsilon \cdot D^{s-2}M^\epsilon dx\right| + \left|\int_{\mathbb{R}} (v^\epsilon)^p \partial_x (D^{s-2}M^\epsilon)^2 dx\right| \\ &\lesssim \|[D^{s-2}\partial_x, (v^\epsilon)^p]M^\epsilon\|_{L^2}\|D^{s-2}M^\epsilon\|_{L^2} + \|\partial_x(v^\epsilon)^p\|_{L^\infty}\|M^\epsilon\|_{H^{s-2}}^2 \\ &\lesssim \|v^\epsilon\|_{H^{s-1}}^p \|M^\epsilon\|_{H^{s-2}}^2, \end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}} D^{s-2}M^\epsilon D^{s-2}\partial_x\{[(v^\epsilon)^p-(v_k^\epsilon)^p]m_k^\epsilon\}dx\right| \\
&\lesssim \left|\int_{\mathbb{R}}[D^{s-2}\partial_x,(v^\epsilon)^p-(v_k^\epsilon)^p]m_k^\epsilon dx\right| \|M^\epsilon\|_{H^{s-2}} + \left|\int_{\mathbb{R}}[(v^\epsilon)^p-(v_k^\epsilon)^p]\partial_x D^{s-2}m_k^\epsilon dx\right| \|M^\epsilon\|_{H^{s-2}} \\
&\lesssim \|[D^{s-2}\partial_x,(v^\epsilon)^p-(v_k^\epsilon)^p]m_k^\epsilon\|_{L^2}\|M^\epsilon\|_{H^{s-2}} + \|\partial_x[(v^\epsilon)^p-(v_k^\epsilon)^p]\|_{L^\infty}\|m_k^\epsilon\|_{H^{s-2}}\|M^\epsilon\|_{H^{s-2}} \\
&\lesssim \|(v^\epsilon)^p-(v_k^\epsilon)^p\|_{H^{s-1}}\|m_k^\epsilon\|_{H^{s-2}}\|M^\epsilon\|_{H^{s-2}},
\end{aligned}$$

where we use Kato–Ponce Lemma 2.2(i), algebra property Lemma 2.1(iii), and the Sobolev's inequality with $s-1>3/2$. Therefore, the last term on the right-hand side of Eq. (2.29) yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}} D^{s-2}M^\epsilon D^{s-2}\left(\frac{p-a}{p}\partial_x(v^\epsilon)^pM^\epsilon + \frac{p-a}{p}\partial_x[(v^\epsilon)^p-(v_k^\epsilon)^p]m_k^\epsilon\right)dx\right| \\
&\quad\lesssim \|\partial_x(v^\epsilon)^pM^\epsilon\|_{H^{s-2}}\|M^\epsilon\|_{H^{s-2}} + \|\partial_x[(v^\epsilon)^p-(v_k^\epsilon)^p]m_k^\epsilon\|_{H^{s-2}}\|M^\epsilon\|_{H^{s-2}} \\
&\quad\lesssim \|v^\epsilon\|_{H^{s-1}}^p\|M^\epsilon\|_{H^{s-2}}^2 + \|(v^\epsilon)^p-(v_k^\epsilon)^p\|_{H^{s-1}}\|m_k^\epsilon\|_{H^{s-2}}\|M^\epsilon\|_{H^{s-2}} \\
&\quad\lesssim \|v^\epsilon\|_{H^{s-1}}^p\|M^\epsilon\|_{H^{s-2}}^2 + \|M^\epsilon\|_{H^{s-2}}\|N^\epsilon\|_{H^{s-1}}\|u_k^\epsilon\|_{H^s}\sum_{i=0}^{p-1}\|v^\epsilon\|_{H^{s-1}}^{p-1-i}\|v_k^\epsilon\|_{H^{s-1}}^i,
\end{aligned}$$

where Lemma 2.1(ii) is applied as $s-2>1/2$. Since Z^ϵ and Z_k^ϵ satisfy the estimate $\|z_k^\epsilon\|_{H^s},\|z^\epsilon\|_{H^s}\lesssim\|z_0\|_{H^s\times H^s}\lesssim 1$ (see (2.11)) and $\|M^\epsilon\|_{H^{s-2}}\cong\|U^\epsilon\|_{H^s},\|N^\epsilon\|_{H^{s-2}}\cong\|V^\epsilon\|_{H^s}$, we may derive

$$\frac{d}{dt}\|U^\epsilon\|_{H^s}\lesssim C_s\left(\|U^\epsilon\|_{H^s}+\|V^\epsilon\|_{H^s}\right).$$

A similar approach could lead to the following result:

$$\frac{d}{dt}\left(\|U^\epsilon\|_{H^s}+\|V^\epsilon\|_{H^s}\right)\le C_s\left(\|U^\epsilon\|_{H^s}+\|V^\epsilon\|_{H^s}\right),$$

which can be solved for all $t\in[0,T]$ with T defined by Eq. (1.3) to generate

$$\|U^\epsilon(t)\|_{H^s}+\|V^\epsilon(t)\|_{H^s}\le\left(\|U^\epsilon(0)\|_{H^s}+\|V^\epsilon(0)\|_{H^s}\right)\exp(C_sT).$$

After ϵ is chosen, we take N sufficiently large so that

$$\|U^\epsilon(0)\|_{H^s}+\|V^\epsilon(0)\|_{H^s}<\frac{\eta}{3}\exp(-C_sT).$$

Therefore, we have $\|z^\epsilon-z_k^\epsilon\|_{\mathcal{C}(I;H^s)}<\eta/3$. Adopting the same approach as above, we can get $\|z_k-z_k^\epsilon\|_{\mathcal{C}(I;H^s)}$, $\|z^\epsilon-z\|_{\mathcal{C}(I;H^s)}<\eta/3$. Thus, the continuity of the data-to-solution map for Eq. (1.1) has been proved.

2.5. Well-posedness on a circle

If the initial data $(u_0,v_0)\in H^s(\mathbb{T}), s>5/2$ is given on a circle, then the CCCH system can be dealt with a approach similar to the line (or nonperiodic) problem with a few modifications. We need a construction of the mollifier j_ϵ and may begin with a function $j\in\mathcal{S}(\mathbb{R})$ and the periodic functions j_ϵ by $j_\epsilon\doteq\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\hat{j}(\epsilon n)e^{inx}$, which admit an analogous estimate to (2.2). Then, the subsequent procedure makes us observe that the proof of existence, uniqueness and continuous dependence can be replicated without any difficulty.

3. Nonuniform dependence of the strong solution to Eq. (1.1)

3.1. Approximate solutions

Let us first construct a two-parameter family of approximate solutions through using a similar method to [25,33], then estimate the error and at last the H^r-norm error. The approximate solution $u^{\omega,\lambda}=u^{\omega,\lambda}(t,x)$

and $v^{\omega,\lambda} = v^{\omega,\lambda}(t,x)$ to (1.1) will consist of a low and a high frequency parts, i.e.,

$$u^{\omega,\lambda} = u_l + u_h, \;\; v^{\omega,\lambda} = v_l + v_h,$$

where ω is a bounded constant, and $\lambda > 0$. The high frequency part is given by

$$u_h = u^{h,\omega,\lambda} = \lambda^{-\frac{\delta}{2p}-s}\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\cos(\lambda x - \omega^p t), v_h = v^{h,\omega,\lambda} = \lambda^{-\frac{\delta}{2q}-s}\varphi\left(\frac{x}{\lambda^{\delta/q}}\right)\cos(\lambda x - \omega^q t), \tag{3.1}$$

with the cutoff function $\phi, \varphi \in C^\infty$ satisfying

$$\phi(x), \varphi(x) = \begin{cases} 1, \text{ if } |x| < 1, \\ 0, \text{ if } |x| \geq 2. \end{cases}$$

Simultaneously, the low frequency parts $u_l = u_{l,\omega,\lambda}(t,x)$ and $v_l = v_{l,\omega,\lambda}(t,x)$ are the solution to Eq. (1.1) with the following initial data

$$\begin{cases} \partial_t u_l + v_l^p \partial_x u_l + D^{-2}\left[\frac{a}{p}u_l\partial_x v_l^p + \frac{p-a}{p}\partial_x v_l^p \partial_x^2 u_l\right] + D^{-2}\partial_x[\partial_x v_l^p \partial_x u_l] = 0, & t \in \mathbb{R}, x \in \mathbb{R}, \\ \partial_t v_l + u_l^q \partial_x v_l + D^{-2}\left[\frac{b}{q}v_l\partial_x u_l^q + \frac{q-b}{q}\partial_x u_l^q \partial_x^2 v_l\right] + D^{-2}\partial_x[\partial_x u_l^q \partial_x v_l] = 0, & t \in \mathbb{R}, x \in \mathbb{R}, \\ u_l(0,x) = \omega\lambda^{-\frac{1}{q}}\tilde{\phi}\left(\frac{x}{\lambda^{\frac{\delta}{q}}}\right), v_l(0,x) = \omega\lambda^{-\frac{1}{p}}\tilde{\varphi}\left(\frac{x}{\lambda^{\frac{\delta}{p}}}\right), & t = 0, x \in \mathbb{R}, \end{cases} \tag{3.2}$$

where $\tilde{\phi}, \tilde{\varphi} \in C_0^\infty$ satisfies

$$\tilde{\phi}^q(x) = 1, \quad \tilde{\varphi}^p(x) = 1 \text{ if } x \in \text{supp}\varphi \cup \text{supp}\phi. \tag{3.3}$$

Let us now study properties of (u_l, v_l) and (u_h, v_h). The high frequency part (u_h, v_h) satisfies

$$\|u_h(t)\|_{L^\infty} \lesssim \lambda^{-\frac{\delta}{2p}-s}, \;\; \|v_h(t)\|_{L^\infty} \lesssim \lambda^{-\frac{\delta}{2q}-s},$$
$$\|\partial_x u_h(t)\|_{L^\infty} \lesssim \lambda^{-\frac{\delta}{2p}-s+1}, \;\; \|\partial_x v_h(t)\|_{L^\infty} \lesssim \lambda^{-\frac{\delta}{2q}-s+1}.$$

To estimate $\|u_h(t)\|_{H^r}$ and $\|v_h(t)\|_{H^r}$, we need the following results.

Lemma 3.1 (*See [25]*)**.** *Let $\psi \in \mathcal{S}(\mathbb{R})$, $\delta > 0$ and $\alpha \in \mathbb{R}$. Then for any $s \geq 0$ we have*

$$\lim_{\lambda\to\infty} \lambda^{-\frac{1}{2}\delta - s}\left\|\psi\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x - \alpha)\right\|_{H^s} = \frac{1}{\sqrt{2}}\|\psi\|_{L^2}. \tag{3.4}$$

Relation (3.4) is also true if cos *is replaced by* sin.

So, this Lemma tells us

$$\|u_h(t)\|_{H^r} \lesssim \lambda^{r-s}\lambda^{-\frac{\delta}{2p}-r}\left\|\psi\left(\frac{x}{\lambda^{\delta/p}}\right)\cos(\lambda x - \alpha)\right\|_{H^r} \lesssim \lambda^{r-s}, \quad \|v_h(t)\|_{H^s} \lesssim \lambda^{r-s}, \text{ for } \lambda \gg 1.$$

Obviously, the low frequency part $(u_l, v_l) = 0$ under the zero initial condition of Eq. (3.2) with $\omega = 0$. For $\omega \neq 0$, but bounded, basic properties of (u_l, v_l) are summarized in the following lemma.

Lemma 3.2. *Let ω be bounded, $0 < \delta < 2$, and $\lambda \gg 1$. Then the initial-value problem (3.2) has a unique solution $(u_l, v_l) \in \mathcal{C}([0,T]; H^s(\mathbb{R})) \times \mathcal{C}([0,T]; H^s(\mathbb{R}))$ with $s > 5/2$. Moreover, for all $r \geq 0$ this solution satisfies the following estimate*

$$\|u_l(t)\|_{H^r} \leq c_r\lambda^{\frac{\delta-2}{2q}}, \|v_l(t)\|_{H^r} \leq c_r\lambda^{\frac{\delta-2}{2p}}, \quad 0 \leq t \leq 1. \tag{3.5}$$

Proof. Apparently, for any function $\phi \in \mathcal{S}(\mathbb{R})$ we have

$$\left\|\phi\left(\frac{x}{\lambda^{k\delta}}\right)\right\|_{H^r} \leq \lambda^{k\delta/2}\left\|\phi\right\|_{H^r}. \tag{3.6}$$

In fact, as per the relation $\widehat{\phi\left(\frac{x}{\rho}\right)}(\xi) = \rho\widehat{\phi}(\rho\xi)$, making the change of variables $\eta = \lambda^{k\delta}\xi$ leads to

$$\begin{aligned}
\left\|\phi\left(\frac{x}{\lambda^{k\delta}}\right)\right\|_{H^r}^2 &= \frac{1}{2\pi}\int_{\mathbb{R}}(1+\xi^2)^r\left|\lambda^{k\delta}\widehat{\phi}(\lambda^{k\delta})\right|^2 d\xi \\
&= \frac{\lambda^{k\delta}}{2\pi}\int_{\mathbb{R}}\left(1+\frac{\eta^2}{\lambda^{2k\delta}}\right)^r\left|\widehat{\phi}(\eta)\right|^2 d\eta \\
&= \frac{\lambda^{k\delta}}{2\pi}\int_{\mathbb{R}}\left(1+\eta^2\right)^r\left|\widehat{\phi}(\eta)\right|^2 d\eta \\
&= \lambda^{k\delta}\left\|\phi\right\|_{H^r}^2.
\end{aligned}$$

Therefore, from the inequality (3.6) we know that the initial data $u_l(0), v_l(0)$ satisfy the following estimate

$$\|u_l(0)\|_{H^r} \leq |\omega|\lambda^{\frac{\delta-2}{2q}}\left\|\tilde{\phi}\right\|_{H^r}$$

which decays if $\delta \leq 2$ and ω is bounded. Furthermore, using the estimate (1.3) from Theorem 1.1, we obtain the lifespan $T \geq \frac{2^\kappa-1}{2^{\kappa+1}\kappa C_s\|z_0\|_{H^s}^\kappa} \geq 1$ for $\lambda \gg 1$ and $\delta \leq 2$. So, if $r \geq 0$ then the estimate (1.4) of Theorem 1.1 yields

$$\|u_l(t)\|_{H^r} \leq \|u_l(t)\|_{H^{r+3}} \leq C_s\|u_l(0)\|_{H^{r+3}} \leq C_s\lambda^{\frac{\delta-2}{2q}}.$$

This completes Lemma 3.2. □

Substituting the approximate solutions $(u^{\omega,\lambda}, v^{\omega,\lambda})$ into Eq. (1.1), and noticing that (u_l, v_l) is a solution to Eq. (3.2), we obtain the following error

$$\begin{aligned}
F(u^{\omega,\lambda}, v^{\omega,\lambda}) &= F_1 + F_2 + F_3 + F_4 + F_5, \\
\tilde{F}(u^{\omega,\lambda}, v^{\omega,\lambda}) &= \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3 + \tilde{F}_4 + \tilde{F}_5,
\end{aligned}$$

where

$$\begin{aligned}
F_1 &= \partial_t u_h + v_l^p\partial_x u_h, \\
F_2 &= \left(\sum_{j=1}^p C_j v_l^{p-j}v_h^j\right)\partial_x(u_l+u_h), \\
F_3 &= \frac{p-a}{p}D^{-2}\left[\partial_x^2(u_l+u_h)\partial_x\left(\sum_{j=1}^p C_j v_l^{p-j}v_h^j\right) + \partial_x^2 u_h\partial_x v_l^p\right], \\
F_4 &= \frac{a}{p}D^{-2}\left[(u_l+u_h)\partial_x\left(\sum_{j=1}^p C_j v_l^{p-j}v_h^j\right) + u_h\partial_x v_l^p\right], \\
F_5 &= D^{-2}\partial_x\left[\partial_x(u_l+u_h)\partial_x\left(\sum_{j=1}^p C_j v_l^{p-j}v_h^j\right) + \partial_x u_h\partial_x v_l^p\right].
\end{aligned}$$

3.2. Estimation of the error F in the H^θ-norm

For our convenience, let us just focus on the estimates for the case $r \geq 0$ and $1 < \delta < 2$:

$$\|u_l(t)\|_{H^r} \lesssim \lambda^{\frac{\delta-2}{2q}}, \ \ \|v_l(t)\|_{H^r} \lesssim \lambda^{\frac{\delta-2}{2p}},$$
$$\|u_h(t)\|_{L^\infty} \lesssim \lambda^{-\frac{\delta}{2p}-s}, \ \ \|v_h(t)\|_{L^\infty} \lesssim \lambda^{-\frac{\delta}{2q}-s},$$
$$\|\partial_x u_h(t)\|_{L^\infty} \lesssim \lambda^{-\frac{\delta}{2p}-s+1}, \ \ \|\partial_x v_h(t)\|_{L^\infty} \lesssim \lambda^{-\frac{\delta}{2q}-s+1},$$
$$\|u_h(t)\|_{H^r}, \ \ \|v_h(t)\|_{H^r} \lesssim \lambda^{-s+r}.$$

If $\theta > 1/2$, by the Sobolev's lemma, Lemma 2.1(i) presents the following algebraic property

$$\|fg\|_{H^\theta} \leq c_\theta \|f\|_{\mathcal{C}^1} \|g\|_{H^\theta}.$$

Next, let us estimate the error F in the H^θ norm where $\theta \in (3/2, s)$.

Estimation of F_1 in the H^θ-norm Apparently, $\tilde{\varphi}^p\left(\frac{x}{\lambda^{\delta/p}}\right)\phi\left(\frac{x}{\lambda^{\delta/p}}\right) = \phi\left(\frac{x}{\lambda^{\delta/p}}\right)$ and $\partial_t u_h$ can be rewritten as

$$\begin{aligned}\partial_t u_h(x,t) &= \omega^p \lambda^{-\frac{\delta}{2p}-s} \tilde{\varphi}^p\left(\frac{x}{\lambda^{\delta/p}}\right)\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\sin(\lambda x - \omega^p t)\\ &= \lambda^{1-\frac{\delta}{2p}-s} v_l^p(x,0)\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\sin(\lambda x - \omega^p t),\end{aligned}$$

and

$$\partial_x u_h(x,t) = -\lambda^{1-\frac{\delta}{2p}-s}\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\sin(\lambda x - \omega^p t) + \lambda^{-\frac{3\delta}{2p}-s}\phi'\left(\frac{x}{\lambda^{\delta/p}}\right)\cos(\lambda x - \omega^p t).$$

Furthermore, we have

$$F_1 = -[v_l^p(t,x) - v_l^p(0,x)]\lambda^{1-\frac{\delta}{2p}-s}\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\sin(\lambda x - \omega^p t) \tag{*}$$
$$+ [v_l^p(t,x)]\lambda^{-\frac{3\delta}{2p}-s}\phi'\left(\frac{x}{\lambda^{\delta/p}}\right)\cos(\lambda x - \omega^p t). \tag{**}$$

Applying the algebraic property leads to

$$\begin{aligned}\|(*)\|_{H^\theta} &\lesssim \lambda^{1-\frac{\delta}{2p}-s}\left\|[v_l^p(t) - v_l^p(0)]\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\sin(\lambda x - \omega^p t)\right\|_{H^\theta}\\ &\lesssim \lambda^{1-\frac{\delta}{2p}-s}\cdot\|v_l^p(t) - v_l^p(0)\|_{H^\theta}\left\|\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\sin(\lambda x - \omega^p t)\right\|_{H^\theta}\\ &\lesssim \lambda^{1-\frac{\delta}{2p}-s}\lambda^{\theta+\delta/2p}\|v_l^p(t) - v_l^p(0)\|_{H^\theta}\\ &\lesssim \lambda^{1-s+\theta}\cdot\|v_l^p(t) - v_l^p(0)\|_{H^\theta}.\end{aligned}$$

To estimate the H^θ-norm of the difference $v_l^p(t) - v_l^p(0)$, we adopt the fundamental theorem of calculus in time variable to obtain

$$\|v_l^p(t) - v_l^p(0)\|_{H^\theta} \leq \int_0^t \left\|v_l^{p-1}(x,\tau)\right\|_{H^\theta}\|\partial_t v_l(x,\tau)\|_{H^\theta}\,d\tau, t \in [0,T].$$

Thus, it follows from Eq. (3.2) that

$$\begin{aligned}\|\partial_t v_l\|_{H^\theta} &\lesssim \|u_l^q\partial_x v_l\|_{H^\theta} + \|v_l\partial_x u_l^q\|_{H^{\theta-2}} + \left\|\partial_x u_l^q\partial_x^2 v_l\right\|_{H^{\theta-2}} + \|\partial_x u_l^q\partial_x v_l\|_{H^{\theta-1}}\\ &\lesssim \|u_l\|_{H^{s+1}}^q\|v_l\|_{H^{s+1}}.\end{aligned}$$

Therefore, we have

$$\|(*)\|_{H^\theta} \lesssim \lambda^{1-s+\theta} \|u_l\|_{H^{s+1}}^q \|v_l\|_{H^{s+1}}^p \lesssim \lambda^{-1-s+\delta+\theta}, \quad \lambda \gg 1.$$

On the other hand, we know

$$\begin{aligned}
\|(**)\|_{H^\theta} &= \left\| v_l^p(x,t)\lambda^{-\frac{3\delta}{2p}-s}\partial_x\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\cos(\lambda x-\omega^p t)\right\|_{H^\theta} \\
&\lesssim \lambda^{-\frac{3\delta}{2p}-s} \|v_l^p(x,t)\|_{H^\theta} \left\|\phi'\left(\frac{x}{\lambda^{\delta/p}}\right)\cos(\lambda x-\omega^p t)\right\|_{H^\theta} \\
&\lesssim \lambda^{-\frac{3\delta}{2p}-s}\lambda^{\theta+\delta/2p}\lambda^{(\delta-2)/2} \\
&\lesssim \lambda^{-s+\delta/2-\delta/p-1+\theta}, \quad \lambda \gg 1.
\end{aligned}$$

Hence, we obtain

$$\|F_1\|_{H^\theta} \lesssim \|(*)\|_{H^\theta} + \|(**)\|_{H^\theta} \lesssim \lambda^{-1-s+\delta+\theta}, \quad \lambda \gg 1.$$

Estimation of F_2 in the H^θ-norm is given by

$$\begin{aligned}
\|F_2\|_{H^\theta} &= \left\| \left(\sum_{j=1}^p C_j v_l^{p-j} v_h^j\right)\partial_x(u_l+u_h)\right\|_{H^\theta} \\
&\lesssim \left\| \left(\sum_{j=1}^p C_j v_l^{p-j} v_h^j\right)\partial_x u_l\right\|_{H^\theta} + \left\| \sum_{j=1}^p C_j v_l^{p-j} v_h^j \partial_x u_h\right\|_{H^\theta} \\
&\lesssim \sum_{j=1}^p \|v_l\|_{H^\theta}^{p-j} \|v_h\|_{H^\theta}^j \|\partial_x u_l\|_{H^\theta} + \sum_{j=1}^p \|v_l\|_{H^\theta}^{p-j} \left(\|v_h\|_{H^\theta}^j \|\partial_x u_h\|_{L^\infty} + \|v_h\|_{L^\infty}^j \|\partial_x u_h\|_{H^\theta}\right) \\
&\lesssim \sum_{j=1}^p \lambda^{\frac{(p+1-j)(\delta-2)}{2p}+j(-s+\theta)} + \sum_{j=1}^p \lambda^{\frac{(p-j)(\delta-2)}{2p}+j(-s+\theta)-\frac{\delta}{2p}-s+1} + \sum_{j=1}^p \lambda^{\frac{(p-j)(\delta-2)}{2p}-j(s+\frac{\delta}{2p})-s+\theta+1} \\
&\lesssim \sum_{j=1}^p \lambda^{j(-s+\theta)} + \sum_{j=1}^p \lambda^{j(-s+\theta)-\frac{\delta}{2p}-s+1} + \sum_{j=1}^p \lambda^{-j(s+\frac{\delta}{2p})-s+\theta+1} \\
&\lesssim \lambda^{-s+\theta} + \lambda^{-2s+\theta-\frac{\delta}{2p}+1} + \lambda^{-2s-\frac{\delta}{2p}+\theta+1}.
\end{aligned}$$

Estimation of F_3 in the H^θ-norm. For $\theta - 1 > 1/2$, it follows from Lemma 2.1(ii) that

$$\begin{aligned}
\|F_3\|_{H^\theta} &= \left\|\partial_x^2(u_l+u_h)\right\|_{H^{\theta-2}} \left\|\partial_x\left(\sum_{j=1}^p C_j v_l^{p-j} v_h^j\right)\right\|_{H^{\theta-1}} + \left\|\partial_x^2 u_h\right\|_{H^{\theta-2}} \|\partial_x v_l^p\|_{H^{\theta-1}} \\
&\lesssim (\|u_l\|_{H^\theta} + \|u_h\|_{H^\theta}) \sum_{j=1}^p \|v_l\|_{H^\theta}^{p-j} \|v_h\|_{H^\theta}^j + \|u_h\|_{H^\theta} \|v_l\|_{H^\theta}^p \\
&\lesssim (\lambda^{\frac{\delta-2}{2p}} + \lambda^{-s+\theta}) \sum_{j=1}^p \lambda^{\frac{(p-j)(\delta-2)}{2p}+j(\theta-s)} + \lambda^{-s+\theta+\frac{p(\delta-2)}{2p}}.
\end{aligned}$$

Estimation of F_4 in the H^θ-norm. For $\theta - 1 > 1/2$, it follows from Lemma 2.1 that

$$\begin{aligned}
\|F_4\|_{H^\theta} &= \left\| (u_l + u_h)\partial_x \left(\sum_{j=1}^{p} C_j v_l^{p-j} v_h^j \right) + u_h \partial_x v_l^p \right\|_{H^{\theta-2}} \\
&\le \|(u_l + u_h)\|_{H^{\theta-1}} \left\| \sum_{j=1}^{p} C_j v_l^{p-j} v_h^j \right\|_{H^{\theta-1}} + \|u_h\|_{H^{\theta-1}} \|v_l^p\|_{H^{\theta-1}} \\
&\lesssim (\|u_l\|_{H^{\theta-1}} + \|u_h\|_{H^{\theta-1}}) \sum_{j=1}^{p} \|v_l\|_{H^{\theta-1}}^{p-j} \|v_h\|_{H^{\theta-1}}^j + \|u_h\|_{H^{\theta-1}} \|v_l\|_{H^{\theta-1}}^p \\
&\lesssim (\|u_l\|_{H^\theta} + \|u_h\|_{H^\theta}) \sum_{j=1}^{p} \|v_l\|_{H^\theta}^{p-j} \|v_h\|_{H^\theta}^j + \|u_h\|_{H^\theta} \|v_l\|_{H^\theta}^p \\
&\lesssim (\lambda^{\frac{\delta-2}{2p}} + \lambda^{-s+\theta}) \sum_{j=1}^{p} \lambda^{\frac{(p-j)(\delta-2)}{2p} + j(\theta-s)} + \lambda^{-s+\theta+\frac{p(\delta-2)}{2p}}.
\end{aligned}$$

Estimation of F_5 in the H^θ-norm. For $\theta - 1 > 1/2$, it follows from Lemma 2.1 that

$$\begin{aligned}
\|F_5\|_{H^\theta} &= \left\| \partial_x (u_l + u_h)\partial_x \left(\sum_{j=1}^{p} C_j v_l^{p-j} v_h^j \right) + \partial_x u_h \partial_x v_l^p \right\|_{H^{\theta-1}} \\
&\lesssim (\|u_l\|_{H^\theta} + \|u_h\|_{H^\theta}) \sum_{j=1}^{p} \|v_l\|_{H^\theta}^{p-j} \|v_h\|_{H^\theta}^j + \|u_h\|_{H^\theta} \|v_l\|_{H^\theta}^p \\
&\lesssim (\lambda^{\frac{\delta-2}{2p}} + \lambda^{-s+\theta}) \sum_{j=1}^{p} \lambda^{\frac{(p-j)(\delta-2)}{2p} + j(\theta-s)} + \lambda^{-s+\theta+\frac{p(\delta-2)}{2p}}.
\end{aligned}$$

Collecting all error estimates together produces the following theorem.

Theorem 3.1. *Let $s > 5/2$, $3/2 < \theta < s$, and $0 < \delta < \min\{2, 1 + s - \theta\}$. If ω is bounded in $\mathbb{R}$ and $\lambda \gg 1$, then we have*

$$\|F\|_{H^\theta}, \|\tilde{F}\|_{H^\theta} \lesssim \lambda^{-\theta_s}, \text{ for } \lambda \gg 1, 0 < t < T, \tag{3.7}$$

where

$$\theta_s = 1 + s - \delta - \theta > 0. \tag{3.8}$$

3.3. Estimation between approximate and actual solutions

Let us now estimate the difference between the approximate and actual solutions. Let $z_{\omega,\lambda}(t,x) = (u_{\omega,\lambda}(t,x), v_{\omega,\lambda}(t,x))$ be the solution to Eq. (1.1) with initial data given by the approximate solution $z^{\omega,\lambda}(t,x) = (u^{\omega,\lambda}(t,x), v^{\omega,\lambda}(t,x))$ evaluated at zero time, that is, $z_{\omega,\lambda}(t,x)$ satisfies

$$\begin{cases}
\partial_t u_{\omega,\lambda} + v_{\omega,\lambda}^p \partial_x u_{\omega,\lambda} + D^{-2}\left[\dfrac{a}{p} u_{\omega,\lambda} \partial_x v_{\omega,\lambda}^p + \dfrac{p-a}{p} \partial_x v_{\omega,\lambda}^p \partial_x^2 u_{\omega,\lambda}\right] + D^{-2}\partial_x[\partial_x v_{\omega,\lambda}^p \partial_x u_{\omega,\lambda}] = 0, \\
\partial_t v_{\omega,\lambda} + u_{\omega,\lambda}^q \partial_x v_{\omega,\lambda} + D^{-2}\left[\dfrac{b}{q} v_{\omega,\lambda} \partial_x u_{\omega,\lambda}^q + \dfrac{q-b}{q} \partial_x u_{\omega,\lambda}^q \partial_x^2 v_{\omega,\lambda}\right] + D^{-2}\partial_x[\partial_x u_{\omega,\lambda}^q \partial_x v_{\omega,\lambda}] = 0, \\
u_{\omega,\lambda}(0,x) = u^{\omega,\lambda}(0,x) = \omega\lambda^{-\frac{1}{q}} \tilde{\phi}\left(\dfrac{x}{\lambda^{\delta/q}}\right) + \lambda^{-\frac{\delta}{2p}-s} \phi\left(\dfrac{x}{\lambda^{\delta/p}}\right) \cos(\lambda x), \\
v_{\omega,\lambda}(0,x) = v^{\omega,\lambda}(0,x) = \omega\lambda^{-\frac{1}{p}} \tilde{\phi}\left(\dfrac{x}{\lambda^{\delta/p}}\right) + \lambda^{-\frac{\delta}{2q}-s} \phi\left(\dfrac{x}{\lambda^{\delta/q}}\right) \cos(\lambda x).
\end{cases} \tag{3.9}$$

Noticing that $z_{\omega,\lambda}(0,x) = (u_{\omega,\lambda}(0,x), v_{\omega,\lambda}(0,x)) \in H^s \times H^s, s > 0$, it follows from Lemmas 3.1 and 3.2 that

$$\begin{aligned} \|u_{\omega,\lambda}(t,x)\|_{H^s} \lesssim \|u_{\omega,\lambda}(0,x)\|_{H^s} \le \|u_l(0)\|_{H^s} + \|u_h(0)\|_{H^s} \lesssim \lambda^{\frac{\delta-2}{2q}} + 1 \lesssim 1, \lambda \gg 1, \\ \|v_{\omega,\lambda}(t,x)\|_{H^s} \lesssim \|v_{\omega,\lambda}(0,x)\|_{H^s} \le \|v_l(0)\|_{H^s} + \|v_h(0)\|_{H^s} \lesssim \lambda^{\frac{\delta-2}{2p}} + 1 \lesssim 1, \lambda \gg 1. \end{aligned} \tag{3.10}$$

Thus, by Theorem 1.1 with $s > 5/2$ we have that for any bounded ω and $\lambda \gg 1$, (3.9) has a unique solution $z_{\omega,\lambda} \in \mathcal{C}([0,T]; H^s) \times \mathcal{C}([0,T]; H^s)$ where

$$T \gtrsim \frac{2^\kappa - 1}{2^{\kappa+1}\kappa C_s \|z_{\omega,\lambda}(0)\|_{H^s}^\kappa} \gtrsim \frac{1}{(\lambda^{\frac{\delta-2}{2\kappa}} + 1)^\kappa}, \lambda \gg 1.$$

Now, let us check if the difference between two sequences of the approximate solutions $z^{\omega,\lambda}(t) - z_{\omega,\lambda}(t)$ goes to zero at time $t = 0$ and stay apart at $t > 0$. Let

$$U = u^{\omega,\lambda} - u_{\omega,\lambda} \doteq u_1 - u_2, V = v^{\omega,\lambda} - v_{\omega,\lambda} \doteq v_1 - v_2.$$

Then, (U, V) solves the following equation

$$\begin{cases} \partial_t U = F(u^{\omega,\lambda}, v^{\omega,\lambda}) - E_1 - E_2 - E_3 - E_4, \\ \partial_t V = \tilde{F}(u^{\omega,\lambda}, v^{\omega,\lambda}) - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - \tilde{E}_4, \\ U(x,0) = V(x,0) = 0, x \in \mathbb{R}, \end{cases} \tag{3.11}$$

where

$$\begin{aligned} E_1 &= v_1^p \partial_x u_1 - v_2^p \partial_x u_2 = v_1^p \partial_x U + \left(\sum_{i=0}^{p-1} v_1^{p-1-i} v_2^i \right) V \partial_x u_2, \\ E_2 &= \frac{p-a}{p} D^{-2} \left[\partial_x v_1^p \partial_x^2 u_1 - \partial_x v_2^p \partial_x^2 u_2 \right] = \frac{p-a}{p} D^{-2} \left[\partial_x^2 U \partial_x v_1^p + \partial_x^2 u_2 \partial_x \left(V \sum_{i=0}^{p-1} v_1^{p-1-i} v_2^i \right) \right], \\ E_3 &= \frac{a}{p} D^{-2} \left[u_1 \partial_x v_1^p - u_1 \partial_x v_1^p \right] = \frac{a}{p} D^{-2} \left[U \partial_x v_1^p + u_2 \partial_x \left(V \sum_{i=0}^{p-1} v_1^{p-1-i} v_2^i \right) \right], \\ E_4 &= D^{-2} \partial_x [\partial_x v_1^p \partial_x u_1 - \partial_x v_2^p \partial_x u_2] = D^{-2} \partial_x \left[\partial_x U \partial_x v_1^p + \partial_x u_2 \partial_x \left(V \sum_{i=0}^{p-1} v_1^{p-1-i} v_2^i \right) \right]. \end{aligned}$$

Proposition 3.1. *Let $s > 5/2$, $3/2 < \theta < s - 1$, and $0 < \delta < \min\{2, 1 + s - \theta\}$. If ω is bounded in $\mathbb{R}$ and $\lambda \gg 1$, then*

$$\|U(t)\|_{H^\theta}, \ \|V(t)\|_{H^\theta} \lesssim \lambda^{-\theta_s}, \ \textit{for } \lambda \gg 1, \ 0 \le t \le T, \tag{3.12}$$

where θ_s is defined by (3.8).

Proof. Plugging the operator D^θ onto both sides of the first equation in (3.11), multiplying by $D^\theta p$, and integrating the resulting equation, we obtain

$$\frac{1}{2} \frac{d}{dt} \|U(t)\|_{H^\theta}^2 = \int_{\mathbb{R}} D^\theta F D^\theta U dx - \int_{\mathbb{R}} D^\theta (E_1 + E_2 + E_3 + E_4) D^\theta U dx. \tag{3.13}$$

Next, let us estimate each term on the right-hand side of (3.13).

Estimation of E_1. As per the Cauchy–Schwarz inequality and Theorem 3.1, we have

$$\left| \int_{\mathbb{R}} D^\theta F D^\theta U dx \right| \le \|F\|_{H^\theta} \|p\|_{H^\theta} \lesssim \lambda^{-\theta_s} \|U(t)\|_{H^\theta}. \tag{3.14}$$

Estimation of E_2. The second term on the right-hand side of (3.13) can be estimated by

$$\begin{aligned}\left|\int_{\mathbb{R}} D^\theta v_1^p \partial_x U D^\theta U dx\right| &\le \left|\int_{\mathbb{R}} D^\theta \left(\partial_x (v_1^p U) - U\partial_x v_1^p\right) D^\theta U dx\right| \\ &\le \left|\int_{\mathbb{R}} [D^\theta \partial_x, v_1^p] U D^\theta U dx\right| + \left|\int_{\mathbb{R}} v_1^p D^\theta \partial_x U D^\theta U dx\right| + \|v_1\|_{H^\theta}^p \|U\|_{H^\theta}^2 \\ &\le \left\|[D^\theta \partial_x, v_1^p] U\right\|_{L^2} \|U\|_{H^\theta} + \frac{1}{2}\left|\int_{\mathbb{R}} \partial_x v_1^p (D^\theta U)^2 dx\right| + \|v_1\|_{H^\theta}^p \|U\|_{H^\theta}^2 \\ &\le (\|v_1\|_{H^\theta}^p \|U\|_{L^\infty} + \|\partial_x v_1^p\|_{L^\infty} \|U\|_{H^\theta}) \|U\|_{H^\theta} + \frac{3}{2}\|v_1\|_{H^\theta}^p \|U\|_{H^\theta}^2 \\ &\lesssim \|v_1\|_{H^\theta}^p \|U\|_{H^\theta}^2.\end{aligned} \tag{3.15}$$

Thus, we get

$$\begin{aligned}\left|\int_{\mathbb{R}} D^\theta E_1 D^\theta U dx\right| &\le \left|\int_{\mathbb{R}} D^\theta v_1^p \partial_x U D^\theta U dx\right| + \left|\int_{\mathbb{R}} D^\theta \left(\sum_{i=0}^{p-1} v_1^{p-1-i} v_2^i\right) V \partial_x u_2 D^\theta U dx\right| \\ &\lesssim \|v_1\|_{H^\theta}^p \|U\|_{H^\theta}^2 + \|U\|_{H^\theta} \|V\|_{H^\theta} \left(\sum_{i=0}^{p-1} \|v_1\|_{H^\theta}^{p-1-i} \|v_2\|_{H^\theta}^i\right) \|u_2\|_{H^{\theta+1}}.\end{aligned} \tag{3.16}$$

If $\theta - 1 > \frac{1}{2}$, by Lemma 2.1 we have

$$\begin{aligned}\left|\int_{\mathbb{R}} D^\theta E_2 D^\theta U dx\right| &\lesssim \left\|\partial_x^2 U \partial_x v_1^p\right\|_{H^{\theta-2}} \|U\|_{H^\theta} + \left\|\partial_x^2 u_2 \partial_x \left(V \sum_{i=0}^{p-1} v_1^{p-1-i} v_2^i\right)\right\|_{H^{\theta-2}} \|U\|_{H^\theta} \\ &\lesssim \left\|\partial_x^2 U\right\|_{H^{\theta-2}} \left\|\partial_x v_1^p\right\|_{H^{\theta-1}} \|U\|_{H^\theta} + \left\|\partial_x^2 u_2\right\|_{H^{\theta-2}} \left\|\partial_x \left(V \sum_{i=0}^{p-1} v_1^{p-1-i} v_2^i\right)\right\|_{H^{\theta-1}} \|U\|_{H^\theta} \\ &\lesssim \left\|v_1^p\right\|_{H^\theta} \|U\|_{H^\theta}^2 + \|V\|_{H^\theta} \|U\|_{H^\theta} \|u_2\|_{H^\theta} \sum_{i=0}^{p-1} \|v_1\|_{H^\theta}^{p-1-i} \|v_2\|_{H^\theta}^i,\end{aligned} \tag{3.17}$$

$$\begin{aligned}\left|\int_{\mathbb{R}} D^\theta E_3 D^\theta U dx\right| &\lesssim \|U \partial_x v_1^p\|_{H^{\theta-1}} \|U\|_{H^\theta} + \left\|u_2 \partial_x \left(V \sum_{i=0}^{p-1} v_1^{p-1-i} v_2^i\right)\right\|_{H^{\theta-1}} \|U\|_{H^\theta} \\ &\lesssim \|v_1^p\|_{H^\theta} \|U\|_{H^\theta}^2 + \|V\|_{H^\theta} \|U\|_{H^\theta} \|u_2\|_{H^\theta} \sum_{i=0}^{p-1} \|v_1\|_{H^\theta}^{p-1-i} \|v_2\|_{H^\theta}^i,\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}} D^\theta E_4 D^\theta U dx\right| &\lesssim \|\partial_x U \partial_x v_1^p\|_{H^{\theta-1}} \|U\|_{H^\theta} + \left\|\partial_x u_2 \partial_x \left(V \sum_{i=0}^{p-1} v_1^{p-1-i} v_2^i\right)\right\|_{H^{\theta-1}} \|U\|_{H^\theta} \\ &\lesssim \|v_1\|_{H^\theta}^p \|U\|_{H^\theta}^2 + \|V\|_{H^\theta} \|U\|_{H^\theta} \|u_2\|_{H^\theta} \sum_{i=0}^{p-1} \|v_1\|_{H^\theta}^{p-1-i} \|v_2\|_{H^\theta}^i.\end{aligned} \tag{3.19}$$

The final differential inequality. Combining the above estimates (3.14)–(3.19) leads to

$$\frac{1}{2}\frac{d}{dt}\|U\|_{H^\theta}^2 \lesssim A_1 \|U\|_{H^\theta}^2 + B_1 \|U\|_{H^\theta} \|V\|_{H^\theta} + \lambda^{-\theta_s} \|U\|_{H^\theta}, \text{ for } \lambda \gg 1,$$

that is

$$\frac{d}{dt}\|U\|_{H^\theta} \lesssim A_1 \|U\|_{H^\theta} + B_1 \|V\|_{H^\theta} + \lambda^{-\theta_s},$$

where $A \lesssim \|v^{\omega,\lambda}\|_{H^\theta}^p = \|v^{\omega,\lambda}\|_{H^s}^p \lesssim 1$, and $B = (\|u_2\|_{H^\theta} + \|u_2\|_{H^{\theta+1}}) \sum_{i=0}^{p-1} \|v_1\|_{H^\theta}^{p-1-i} \|v_2\|_{H^\theta}^i \lesssim 1$. In the above calculation, we utilized the following well-posedness inequality

$$\begin{aligned} \|u_1\|_{H^\theta} = \left\|u^{\omega,\lambda}\right\|_{H^\theta} \le \left\|u^{\omega,\lambda}\right\|_{H^s} \lesssim \lambda^{\frac{\delta-2}{2q}} + 1, \|u_2\|_{H^\theta} = \|u_{\omega,\lambda}\|_{H^\theta} \le \|u_{\omega,\lambda}\|_{H^s} \lesssim \lambda^{\frac{\delta-2}{2q}} + 1. \\ \|v_1\|_{H^\theta} = \left\|v^{\omega,\lambda}\right\|_{H^\theta} \le \left\|v^{\omega,\lambda}\right\|_{H^s} \lesssim \lambda^{\frac{\delta-2}{2p}} + 1, \|v_2\|_{H^\theta} = \|v_{\omega,\lambda}\|_{H^\theta} \le \|v_{\omega,\lambda}\|_{H^s} \lesssim \lambda^{\frac{\delta-2}{2p}} + 1. \end{aligned} \tag{3.20}$$

In a similar way, we are able to get

$$\frac{d}{dt}\|V\|_{H^\theta} \lesssim A_1\|V\|_{H^\theta} + B_1\|U\|_{H^\theta} + \lambda^{-\theta_s}.$$

Therefore, we have

$$\frac{d}{dt}(\|U\|_{H^\theta} + \|V\|_{H^\theta}) \lesssim M(\|U\|_{H^\theta} + \|V\|_{H^\theta}) + \lambda^{-\theta_s},$$

which can be solved with the initial condition $\|U(0)\|_{H^r} = \|V(0)\|_{H^r} = 0$ in the form of

$$\|U(t)\|_{H^r} \lesssim \lambda^{-\theta_s}, \|V(t)\|_{H^r} \lesssim \lambda^{-\theta_s}, \text{ for } \lambda \gg 1, 0 \le t \le T, \tag{3.21}$$

where θ_s is defined by (3.8). This concludes the proof of Proposition 3.1. □

Let us now present the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $s > 5/2$, and assume $(u_{1,\lambda}(t,x), v_{1,\lambda}(t,x))$ and $(\tilde{u}_{0,\lambda}(t,x), \tilde{v}_{0,\lambda}(t,x))$ are the unique solutions to (3.9) with initial data $(u_{1,\lambda}(0,x), v_{1,\lambda}(0,x))$ and $(\tilde{u}_{0,\lambda}(0,x), \tilde{v}_{0,\lambda}(0,x))$, respectively.

It follows from Theorem 1.1 that these solutions belong to $\mathcal{C}([0,T]; H^s) \times \mathcal{C}([0,T]; H^s)$. By (3.10) and the assumptions right after Theorem 1.1, we see that T is independent of $\lambda \gg 1$ and $0 < \delta < \min\{2, 1+s-\theta\}$. Denoting $k = [s] + 2$ and applying estimate (1.4), we have

$$\|u_{\omega,\lambda}(t)\|_{H^k}, \|v_{\omega,\lambda}(t)\|_{H^k} \lesssim \|z^{\omega,\lambda}(0)\|_{H^k} \lesssim \lambda^{k-s}, \text{ for } \lambda \gg 1, 0 \le t \le T. \tag{3.22}$$

By (3.4) and (3.5), we obtain

$$\|u^{\omega,\lambda}(t)\|_{H^k}, \|v^{\omega,\lambda}(t)\|_{H^k} \le \|z_l(t)\|_{H^k} + \|z_h(t)\|_{H^k} \lesssim \lambda^{k-s}. \tag{3.23}$$

So, from (3.22)–(3.23) we have the following estimate for the difference between $z^{\omega,\lambda}$ and $z_{\omega,\lambda}$ in the H^k-norm:

$$\|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^k}, \|v^{\omega,\lambda}(t) - v_{\omega,\lambda}(t)\|_{H^k} \lesssim \lambda^{k-s}, 0 \le t \le T. \tag{3.24}$$

On the other hand, applying (3.12) with the choice of $\omega \in \{0,1\}$ generates

$$\|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^\theta}, \|v^{\omega,\lambda}(t) - v_{\omega,\lambda}(t)\|_{H^\theta} \lesssim \lambda^{-\theta_s}, 0 \le t \le T. \tag{3.25}$$

By the interpolation inequality with $s_1 = \theta$ and $s_2 = [s] + 2 = k$

$$\|f\|_{H^s} \le \|f\|_{H^{s_1}}^{\frac{s_2-s}{s_2-s_1}} \|f\|_{H^{s_2}}^{\frac{s-s_1}{s_2-s_1}}, \tag{3.26}$$

and the estimates (3.24) and (3.25), we obtain

$$\begin{aligned} \|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^s} &\le \|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^\theta}^{\frac{k-s}{k-\theta}} \|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^k}^{\frac{s-\theta}{k-\theta}} \\ &\lesssim \lambda^{\frac{-\theta_s(k-s)}{k-\theta}} \lambda^{\frac{(k-s)(s-\theta)}{k-\theta}} \\ &\lesssim \lambda^{\frac{-(\theta_s-s+\theta)(k-s)}{k-\theta}} \\ &\lesssim \lambda^{\frac{-(1-\delta)(k-s)}{k-\theta}}, \\ \|v^{\omega,\lambda}(t) - v_{\omega,\lambda}(t)\|_{H^s} &\lesssim \lambda^{\frac{-(1-\delta)(k-s)}{k-\theta}}. \end{aligned} \tag{3.27}$$

Obviously, we must have $\frac{(1-\delta)(k-s)}{k-r} > 0$, which is equivalent to $\delta < 1$.

Next, we shall apply the estimate (3.27) to prove nonuniform dependence when $s > 5/2$.

Behavior at $t = 0$. Since $0 < \delta < 1$, at $t = 0$ we have

$$\begin{aligned} &\|u_{1,\lambda}(0) - u_{0,\lambda}(0)\|_{H^s} \lesssim \lambda^{\frac{\delta-2}{2p}} \|\tilde{\phi}\|_{H^s} \to 0 \text{ as } \lambda \to \infty. \\ &\|v_{1,\lambda}(0) - v_{0,\lambda}(0)\|_{H^s} \lesssim \lambda^{\frac{\delta-2}{2q}} \|\tilde{\varphi}\|_{H^s} \to 0 \text{ as } \lambda \to \infty. \end{aligned} \tag{3.28}$$

Behavior at time $t > 0$. Let us write

$$\|u_{1,\lambda}(t) - u_{0,\lambda}(t)\|_{H^s} \geq \|u^{1,\lambda}(t) - u^{0,\lambda}(t)\|_{H^s} - \|u^{1,\lambda}(t) - u_{1,\lambda}(t)\|_{H^s} - \|u^{0,\lambda}(t) - u_{0,\lambda}(t)\|_{H^s}. \tag{3.29}$$

Applying the estimate (3.25) to the last two terms in (3.29) generates

$$\|u_{1,\lambda}(t) - u_{0,\lambda}(t)\|_{H^s} \geq \|u^{1,\lambda}(t) - u^{0,\lambda}(t)\|_{H^s} - c\lambda^{\frac{-(1-\delta)(k-s)}{k-r}}, \tag{3.30}$$

that is

$$\liminf_{\lambda\to\infty} \|u_{1,\lambda}(t) - u_{0,\lambda}(t)\|_{H^s} \geq \liminf_{\lambda\to\infty} \|u^{1,\lambda}(t) - u^{0,\lambda}(t)\|_{H^s}, \tag{3.31}$$

So, it suffices to estimate $\|u^{1,\lambda}(t) - u^{0,\lambda}(t)\|_{H^s}$ below. Since

$$\begin{aligned} u^{1,\lambda}(t) - u^{0,\lambda}(t) &= \lambda^{-\delta/2p-s}\phi\left(\frac{x}{\lambda^{\delta/p}}\right)[\cos(\lambda x - t) - \cos(\lambda x)] + u_{l,1,\lambda}(t) \\ &= 2\lambda^{-\delta/2p-s}\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\sin(\lambda x - \frac{t}{2})\sin\frac{t}{2} + u_{l,1,\lambda}(t), \end{aligned} \tag{3.32}$$

we have

$$\|u^{1,\lambda}(t) - u^{0,\lambda}(t)\|_{H^s} \geq 2\lambda^{-\delta/2p-s}\left\|\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\sin(\lambda x - \frac{t}{2})\right\|_{H^s} |\sin t/2| - c_s\lambda^{(\delta-2)/2p}. \tag{3.33}$$

Thus, using the relation

$$\frac{1}{\sqrt{2}}\|\phi\|_{L^2} = \lim_{\lambda\to\infty} \lambda^{-\delta/2p-s}\left\|\phi\left(\frac{x}{\lambda^{\delta/p}}\right)\sin(\lambda x - \frac{t}{2})\right\|_{H^s}, \tag{3.34}$$

and (3.33), we obtain

$$\liminf_{\lambda\to\infty} \|u^{1,\lambda}(t) - u_{0,\lambda}(t)\|_{H^s} \geq \sqrt{2}\|\phi\|_{L^2}|\sin t/2|. \tag{3.35}$$

Noticing that $|\sin t| = \sin t, 0 \leq t \leq \pi$ gives

$$\liminf_{\lambda\to\infty} \|u_{1,\lambda}(t) - u_{0,\lambda}(t)\|_{H^s} \geq \sqrt{2}\|\phi\|_{L^2}\sin t/2, \tag{3.36}$$

for $0 \leq t < \min\{T, 2\pi\}$.

In short, there exist two sequences of solutions $z_{0,\lambda} = (u_{0,\lambda}(t), v_{0,\lambda}(t))$ and $z_{1,\lambda}(t) = (u_{1,\lambda}(t), v_{1,\lambda}(t))$ to the differential Eq. (1.1) in $\mathcal{C}([0,T]; H^s(\mathbb{R})) \times \mathcal{C}([0,T]; H^s(\mathbb{R}))$ such that

$$\|u_{0,\lambda}(t)\|_{H^s} + \|u_{1,\lambda}(t)\|_{H^s} + \|v_{0,\lambda}(t)\|_{H^s} + \|v_{1,\lambda}(t)\|_{H^s} \lesssim 1, \tag{3.37}$$

$$\lim_{n\to\infty} \|u_{0,\lambda}(0) - u_{1,\lambda}(0)\|_{H^s} = \lim_{n\to\infty} \|v_{0,\lambda}(0) - v_{1,\lambda}(0)\|_{H^s} = 0, \tag{3.38}$$

and

$$\begin{aligned} &\liminf_{n\to\infty} \|u_{0,\lambda}(t) - u_{1,\lambda}(t)\|_{H^s} > 0, 0 < t < \min\{2\pi, T\}, \\ &\liminf_{n\to\infty} \|v_{0,\lambda}(t) - v_{1,\lambda}(t)\|_{H^s} > 0, 0 < t < \min\{2\pi, T\}. \end{aligned}$$

This concludes the proof of Theorem 1.2. □

3.4. Nonuniform dependence on a circle

To prove Theorem 1.2 on a circle, similar to the nonperiodic case, it suffices to show that there exist two sequences of solutions $z = (u_\lambda(t), v_\lambda(t))$ and $\tilde{z}(t) = (\tilde{u}_\lambda(t), \tilde{v}_\lambda(t))$ to the differential Eq. (1.1) in $\mathcal{C}([0,T]; H^s(\mathbb{T})) \times \mathcal{C}([0,T]; H^s(\mathbb{T}))$ such that

$$\|u_\lambda(t)\|_{H^s} + \|\tilde{u}_\lambda(t)\|_{H^s} + \|v_\lambda(t)\|_{H^s} + \|\tilde{v}_\lambda(t)\|_{H^s} \lesssim 1,$$

$$\lim_{n\to\infty} \|u_\lambda(0) - \tilde{u}_\lambda(0)\|_{H^s} = \lim_{n\to\infty} \|v_\lambda(0) - \tilde{v}_\lambda(0)\|_{H^s} = 0,$$

and

$$\liminf_{n\to\infty} \|u_\lambda(0) - \tilde{u}_\lambda(0)\|_{H^s} \gtrsim \sin t,$$
$$\liminf_{n\to\infty} \|v_\lambda(0) - \tilde{v}_\lambda(0)\|_{H^s} \gtrsim \sin t.$$

To see this, we consider the approximate solutions in the form of

$$u^{\omega,\lambda} = \omega\lambda^{-\frac{1}{p}} + \lambda^{-s}\cos(\lambda x - \omega^p t), v^{\omega,\lambda} = \omega\lambda^{-\frac{1}{q}} + \lambda^{-s}\cos(\lambda x - \omega^q t),$$

where $\lambda \in \mathbb{Z}^+$ and $\omega = \pm 1$. Similar to the proof on a line and the proof on a circle for the CH and the 2CH in [26,34], the results listed above for Eq. (1.1) on a circle can be carried out.

4. Hölder continuous in H^r-topology

Theorem 1.1 tells us that the CCCH initial value problem is well-posed for $(u_0, v_0) \in H^s \times H^s, s > 5/2$ with $(u, v) \in \mathcal{C}([0,T]; H^s) \times \mathcal{C}([0,T]; H^s)$. Moreover, in Theorem 1.2 we have shown that the solution map $(u_0, v_0) \in H^s \times H^s \mapsto (u, v) \in \mathcal{C}([0,T]; H^s) \times \mathcal{C}([0,T]; H^s)$ is continuous but not uniformly continuous in $H^s \times H^s$.

Let us now make a further investigation about the continuity properties for the solution map in Hölder spaces $H^r, r < s$ with initial data still in $H^s, s > 5/2$. More precisely, we consider two solutions of Eq. (1.1), $z = (u_1, v_1)$ and $w = (u_2, v_2)$, which emanate from the initial data $z_0 = (u_{0,1}, v_{0,1})$ and $w_0 = (u_{0,2}, v_{0,2})$, respectively. We want to show that if the initial data z_0, w_0 are assigned in a ball with radius ρ in H^s, i.e.,

$$\|z_0\|_{H^s} \le \rho, \|w_0\|_{H^s} \le \rho, s > 5/2, \tag{4.1}$$

then we have

$$\|z(t) - w(t)\|_{H^r} \lesssim \|z_0 - w_0\|_{H^r}^{\alpha}, r < s,$$

where the Hölder exponent α is to be determined.

The proof of Theorem 1.3 is inspired from the work on b-family equation [38], the Novikov equation [39], the FORQ equation [35], and the two-component Camassa–Holm equation [34]. The following three lemmas are needed to prove Theorem 1.3.

Lemma 4.1 (*See [37]*). *If $f \in H^{s-1}$ and $g \in H^\theta$, then*

$$\|[D^\theta \partial_x, f]g\|_{L^2} \le c_{\theta,s}\|f\|_{H^{s-1}}\|g\|_{H^\theta}, \theta + 1 \ge 0, s - 1 > 3/2, \theta + 1 \le s - 1.$$

Lemma 4.2 (*See [35,36]*). *If $\theta > 0$, then $H^\theta \cap L^\infty$ is an algebra. Moreover, we have*
(i) $\|fg\|_{H^\theta} \le c_\theta \|f\|_{H^\theta}\|g\|_{H^\theta}$, *for* $\theta > 1/2$.
(ii) $\|fg\|_{H^\theta} \le c\|f\|_{H^{\theta+1}}\|g\|_{H^\theta}$, *for* $\theta > -1/2$.
(iii) $\|fg\|_{H^\theta} \le c\|f\|_{H^{s-2}}\|g\|_{H^\theta}$, *for* $-1 \le \theta \le 0, s - 1 > 3/2, \theta + s \ge 2$.

Lemma 4.3 (*See [35,36]*). *Suppose $\sigma_1 < \sigma < \sigma_2$ and $f \in H_1^\sigma$. Then, we have*

$$\|f\|_{H^\sigma} \le \|f\|_{H^{\sigma_1}}^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}} \|f\|_{H^{\sigma_2}}^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}},$$

Proof of Theorem 1.3. Differentiating the CCCH system with respect to x, simplifying the resulting equation, and letting $t=-t$, $f=u_x$ and $g=v_x$, we get the following equation

$$\begin{cases} \partial_t u = v^p\partial_x u + I_{11}(u,v,f,g) + I_{12}(u,v,f,g) + \partial_x I_{13}(u,v,f,g), \\ \partial_t v = u^q\partial_x v + I_{21}(u,v,f,g) + I_{22}(u,v,f,g) + \partial_x I_{23}(u,v,f,g), \\ \partial_t f = v^p\partial_x f + \partial_x I_{11}(u,v,f,g) + \partial_x I_{12}(u,v,f,g) + I_{13}(u,v,f,g), \\ \partial_t g = u^q\partial_x g + \partial_x I_{21}(u,v,f,g) + \partial_x I_{22}(u,v,f,g) + I_{23}(u,v,f,g), \\ u_0(x)=u(x,0), v_0(x)=v(x,0), f_0(x)=\partial_x u_0(x) \doteq f_0(x), g_0(x)=\partial_x v_0(x) \doteq g_0(x). \end{cases} \tag{4.2}$$

where

$$\begin{cases} I_{11}(u,v,f,g) = \dfrac{a}{p}(1-\partial_x^2)^{-1}(\partial_x(v^p)u), & I_{21}(u,v,f,g) = \frac{b}{q}(1-\partial_x^2)^{-1}(\partial_x(u^q)v), \\ I_{12}(u,v,f,g) = \dfrac{p-a}{p}(1-\partial_x^2)^{-1}(\partial_x(v^p)\partial_x f), & I_{22}(u,v,f,g) = \frac{q-b}{q}(1-\partial_x^2)^{-1}(\partial_x(u^q)\partial_x g), \\ I_{13}(u,v,f,g) = (1-\partial_x^2)^{-1}(\partial_x(v^p)f), & I_{23}(u,v,f,g) = (1-\partial_x^2)^{-1}(\partial_x(u^q)g). \end{cases}$$

It should be pointed out that in the periodic case the integration is over $\mathbb{T}$. Since all estimates are the same on both the line and the circle, in what follows we shall keep using the notation of the line.

Let us consider two solutions $\phi=(u_1,v_1,f_1,g_1)$ and $\varphi=(u_2,v_2,f_2,g_2)$ to Eq. (4.2), which correspond to the initial data $\phi_0=(u_{0,1},v_{0,1},\partial_x u_{0,1},\partial_x v_{0,1})$ and $\varphi_0=(u_{0,2},v_{0,2},\partial_x u_{0,2},v_{0,2})$, respectively. If the initial data ϕ_0,φ_0 are located in a ball with radius ρ in H^{s-1}, i.e.,

$$\|\phi_0\|_{H^{s-1}} \le \rho, \|\varphi_0\|_{H^{s-1}} \le \rho, s>5/2, \tag{4.3}$$

then, from the estimate (1.4) in Theorem 1.1 we have

$$\|\phi(t)\|_{H^{s-1}} \le \rho, \|\varphi(t)\|_{H^{s-1}} \le \rho, s>5/2. \tag{4.4}$$

Let $U=u_1-u_2, V=v_1-v_2, F=f_1-f_2, G=g_1-g_2$. Then U,V,F,G satisfy the following system

$$\begin{aligned} \partial_t U &= v_1^p\partial_x U + V\partial_x u_2\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i + I_{11}(u_1,v_1,f_1,g_1) - I_{11}(u_2,v_2,f_2,g_2) \\ &\quad + I_{12}(u_1,v_1,f_1,g_1) - I_{12}(u_2,v_2,f_2,g_2) + \partial_x I_{13}(u_1,v_1,f_1,g_1) - \partial_x I_{13}(u_2,v_2,f_2,g_2), \\ \partial_t V &= u_1^q\partial_x V + U\partial_x v_2\sum_{i=0}^{q-1} u_1^{q-1-i}u_2^i + I_{21}(u_1,v_1,f_1,g_1) - I_{21}(u_2,v_2,f_2,g_2) \\ &\quad + I_{22}(u_1,v_1,f_1,g_1) - I_{22}(u_2,v_2,f_2,g_2) + \partial_x I_{23}(u_1,v_1,f_1,g_1) - \partial_x I_{23}(u_2,v_2,f_2,g_2), \\ \partial_t F &= v_1^p\partial_x F + V\partial_x f_2\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i + \partial_x I_{11}(u_1,v_1,f_1,g_1) - \partial_x I_{11}(u_2,v_2,f_2,g_2) \\ &\quad + \partial_x I_{12}(u_1,v_1,f_1,g_1) - \partial_x I_{12}(u_2,v_2,f_2,g_2) + I_{13}(u_1,v_1,f_1,g_1) - I_{13}(u_2,v_2,f_2,g_2), \\ \partial_t G &= u_1^q\partial_x G + U\partial_x g_2\sum_{i=0}^{q-1} u_1^{q-1-i}u_2^i + \partial_x I_{21}(u_1,v_1,f_1,g_1) - \partial_x I_{21}(u_2,v_2,f_2,g_2) \\ &\quad + \partial_x I_{22}(u_1,v_1,f_1,g_1) - \partial_x I_{22}(u_2,v_2,f_2,g_2) + I_{23}(u_1,v_1,f_1,g_1) - I_{23}(u_2,v_2,f_2,g_2), \\ & U_0(x) = u_{0,1}(x) - u_{0,2}(x), V_0(x) = v_{0,1}(x) - v_{0,2}(x), \\ & F_0(x) = \partial_x(u_{0,1}(x) - u_{0,2}(x)), G_0(x) = \partial_x(v_{0,1}(x) - v_{0,2}(x)). \end{aligned} \tag{F}$$

Lipschitz continuity in A_1. We shall show that the solution map for (1.1) is Lipschitz continuous for $(s,r)\in A_1$, that is, the Hölder exponent $\alpha=1$ in the domain A_1. Applying the operator D^r to both sides of equation (F), multiplying by D^rF, and integrating, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|F\|_{H^r}^2 &= \int_{\mathbb{R}} D^r\left(v_1^p\partial_x F\right)D^rF dx + \int_{\mathbb{R}} D^r\left(V\partial_x f_2\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right)D^rF dx\\ &\quad+\int_{\mathbb{R}} D^r(\partial_x I_{11}(u_1,v_1,f_1,g_1)-\partial_x I_{11}(u_2,v_2f_2,v_2))D^rF dx\\ &\quad+\frac{a}{p}\int_{\mathbb{R}} D^r(\partial_x I_{12}(u_1,v_1,f_1,g_1)-\partial_x I_{12}(u_2,v_2f_2,v_2))D^rF dx\\ &\quad+\frac{a}{p}\int_{\mathbb{R}} D^r(I_{13}(u_1,v_1,f_1,g_1)-I_{13}(u_2,v_2f_2,v_2))D^rF dx\\ &\doteq B_1+B_2+B_3+B_4+B_5.\end{aligned}\tag{4.5}$$

We need to estimate the right-hand side of (4.5). Apparently, D^r is commutative.

Estimation of B_1. A direct calculation sends the first term on the right-hand sides of (4.5) to

$$\begin{aligned}|B_1| &= \left|\int_{\mathbb{R}} D^r[\partial_x\left(v_1^pF\right)-F\partial_x v_1^p]D^rF dx\right|\\ &\lesssim \left|\int_{\mathbb{R}}[D^r\partial_x,v_1^p]FD^rF dx\right|+\left|\int_{\mathbb{R}}(v_1^pD^r\partial_xF)D^rF dx\right|+\left|\int_{\mathbb{R}} D^rF\partial_x v_1^pD^rF dx\right|.\end{aligned}\tag{4.6}$$

The first integral can be estimated through the Calderon–Coifman–Meyer commutator described in Lemma 4.1 for $r+1\geq 0, s-1>3/2, r+1\leq s-1$. Employing the algebraic property $\|v_1^p\|_{H^{s-1}}\lesssim\|v_1\|_{H^{s-1}}^p\lesssim\rho^p$ and the Sobolev inequality $\|(v_1^p)_x\|_{L^\infty}\lesssim\|v_1^p\|_{H^{s-1}}\lesssim\rho^p$ yields

$$\left|\int_{\mathbb{R}}[D^r\partial_x,v_1^p]FD^rF dx\right|\lesssim\|[D^r\partial_x,v_1^p]F\|_{L^2}\|F\|_{H^r}\lesssim\|v_1^p\|_{H^{s-1}}\|F\|_{H^r}^2\lesssim\rho^p\|F\|_{H^r}^2.$$

The second integral of (4.6) can be handled through integration by parts and the Sobolev's lemma

$$\left|\int_{\mathbb{R}}(v_1^pD^r\partial_xF)D^rF dx\right|\lesssim\left|\int_{\mathbb{R}}(v_1^p)_x(D^rF)^2dx\right|\lesssim\|(v_1^p)_x\|_{L^\infty}\|F\|_{H^r}^2\lesssim\rho^p\|F\|_{H^r}^2.$$

The third integral of (4.6) would be calculated as follows

$$\begin{aligned}&\left|\int_{\mathbb{R}} D^rF\partial_x v_1^pD^rF dx\right|\lesssim\|F\partial_x v_1^p\|_{H^r}\|F\|_{H^r}\\ &\lesssim\begin{cases}\|v_1^{p-1}g_1\|_{H^r}\|F\|_{H^r}^2\lesssim\|v_1^{p-1}g_1\|_{H^{s-1}}\|F\|_{H^r}^2\lesssim\rho^p\|F\|_{H^r}^2, \text{ for } 1/2<r\leq s-1;\\ \|v_1^{p-1}g_1\|_{H^{r+1}}\|F\|_{H^r}^2\lesssim\|v_1^{p-1}g_1\|_{H^{s-1}}\|F\|_{H^r}^2\lesssim\rho^p\|F\|_{H^r}^2, \text{ for } -1/2<r\leq 1/2, r+2\leq s;\\ \|v_1^{p-1}g_1\|_{H^{s-1}}\|F\|_{H^r}^2\lesssim\rho^p\|F\|_{H^r}^2, \text{ for } -1\leq r\leq -1/2, s-1>3/2, r+s\geq 2.\end{cases}\end{aligned}$$

To sum up, we arrive at

$$|B_1|\lesssim\rho^p\|F\|_{H^r}^2, \text{ for } (r,s)\in\{1/2<r<s-1\}\cup\{-1/2<r\leq 1/2, r+2\leq s\}\cup\{-1\leq r\leq 0, r+s\geq 2\}.$$

Estimation of B_2. A long computation yields the following estimate of B_2:

$$
\begin{aligned}
|B_2| &= \left|\int_{\mathbb{R}} D^r\left(V\partial_x f_2\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right)D^r F dx\right| \lesssim \left\|V\partial_x f_2\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right\|_{H^r}\|F\|_{H^r} \\
&\lesssim \begin{cases}
\left\|\partial_x f_2\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right\|_{H^r}\|V\|_{H^r}\|F\|_{H^r} \lesssim \|f_2\|_{H^{r+1}}\left\|\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right\|_{H^r}\|F\|_{H^r}\|V\|_{H^r} \\
\qquad \lesssim \|f_2\|_{H^{s-1}}\left\|\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right\|_{H^s}\|F\|_{H^r}\|V\|_{H^r} \lesssim \rho^p\|F\|_{H^r}\|V\|_{H^r}, \text{ for } 1/2<r\le s-2; \\
\left\|\partial_x f_2\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right\|_{H^{r+1}}\|V\|_{H^r}\|F\|_{H^r} \lesssim \|f_2\|_{H^{r+2}}\left\|\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right\|_{H^{r+1}}\|F\|_{H^r}\|V\|_{H^r} \\
\qquad \lesssim \|f_2\|_{H^{s-1}}\left\|\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right\|_{H^s}\|F\|_{H^r}\|V\|_{H^r} \lesssim \rho^p\|F\|_{H^r}\|V\|_{H^r}, \text{ for } -1/2<r\le 1/2, r+3\le s; \\
\left\|\partial_x f_2\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right\|_{H^{s-2}}\|V\|_{H^r}\|F\|_{H^r} \lesssim \|f_2\|_{H^{s-1}}\left\|\sum_{i=0}^{p-1} v_1^{p-1-i}v_2^i\right\|_{H^{s-1}}\|F\|_{H^r}\|V\|_{H^r} \\
\qquad \lesssim \rho^p\|F\|_{H^r}\|V\|_{H^r}, \text{ for } -1\le r\le 0, s-1>3/2, r+s\ge 2.
\end{cases} \\
&\lesssim \rho^p\|F\|_{H^r}^2, \text{ for } (r,s)\in\{1/2<r\le s-2\}\cup\{-1/2<r\le 1/2, r+3\le s\}\cup\{-1\le r\le -1/2, r+s\ge 2\}.
\end{aligned}
$$

Estimation of B_4. We shall show the details for the estimate of the most delicate nonlocal term B_4 while other nonlocal terms can be done likewise. A direct calculation generates

$$
\begin{aligned}
|B_4| &\lesssim \|\partial_x(v_1^p)\partial_x f_1 - \partial_x(v_2^p)\partial_x f_2\|_{H^{r-1}}\|F\|_{H^r} \\
&\lesssim (\|\partial_x(v_1^p - v_2^p)\partial_x f_2\|_{H^{r-1}} + \|\partial_x v_1^p\partial_x F\|_{H^{r-1}})\|F\|_{H^r}.
\end{aligned}
$$

Applying Lemma 4.2 yields

$$
\begin{aligned}
\|\partial_x(v_1^p - v_2^p)\partial_x f_2\|_{H^{r-1}} &\lesssim \left\|V g_1\partial_x f_2\sum_{i=0}^{p-2} v_1^{p-2-i}v_2^i\right\|_{H^{r-1}} + \left\|v_2^{p-1}G\partial_x f_2\right\|_{H^{r-1}} \\
&\lesssim \begin{cases}
\left\|g_1\partial_x f_2\sum_{i=0}^{p-2} v_1^{p-2-i}v_2^i\right\|_{H^{r-1}}\|V\|_{H^r} + \left\|v_2^{p-1}\partial_x f_2\right\|_{H^{r-1}}\|G\|_{H^r} \\
\qquad \lesssim \rho^p(\|V\|_{H^r} + \|G\|_{H^r}), \text{ for } 1/2<r\le s-1; \\
\left\|V g_1\partial_x f_2\sum_{i=0}^{p-2} v_1^{p-2-i}v_2^i\right\|_{H^r} + \left\|v_2^{p-1}G\partial_x f_2\right\|_{H^r} \\
\qquad \lesssim \|V\|_{H^r}\|\partial_x f_2 g_1\sum_{i=0}^{p-2} v_1^{p-2-i}v_2^i\|_{H^{s-2}} + \|G\|_{H^r}\|v_2^{p-1}\partial_x f_2\|_{H^{s-2}} \\
\qquad \lesssim \rho^p(\|V\|_{H^r} + \|G\|_{H^r}), \text{ for } -1\le r\le 1/2, s-1>3/2, r+s\ge 2.
\end{cases}
\end{aligned}
$$

Driving a similar proof of Lemma 3 in [39] , we can get $\|fg\|_{H^{r-2}} \le c\|f\|_{H^{s-1}}\|g\|_{H^{r-2}}$ for $0\le r\le 2, s>3/2, r+s\ge 3$. Using this fact and Lemma 4.2, we get

$$
\begin{aligned}
&\|\partial_x v_1^p\partial_x F\|_{H^{r-1}} \\
&\lesssim \begin{cases}
\left\|v_1^{p-1}g\right\|_{H^r}\|\partial_x F\|_{H^{r-1}} \lesssim \left\|v_1^{p-1}g\right\|_{H^{s-1}}\|F\|_{H^r}, \text{ for } 1/2<r\le s-1; \\
\left\|v_1^{p-1}g\right\|_{H^{s-1}}\|\partial_x F\|_{H^{r-1}} \lesssim \left\|v_1^{p-1}g\right\|_{H^{s-1}}\|F\|_{H^r}, \text{ for } -1\le r\le 1/2, s-1>3/2, r+s\ge 2.
\end{cases}
\end{aligned}
$$

Therefore

$$
|B_4| \lesssim \rho^p\|F\|_{H^r}^2, \text{ for } (r,s)\in\{1/2<r\le s-1\}\cup\{-1\le r\le 1/2, s-1>3/2, r+s\ge 2\}.
$$

After estimating other terms in a similar manner, we obtain,

$$\frac{d}{dt}(\|U\|_{H^r}+\|V\|_{H^r}+\|F\|_{H^r}+\|G\|_{H^r})\le \mathcal{C}(\|U\|_{H^r}+\|V\|_{H^r}+\|F\|_{H^r}+\|G\|_{H^r}),$$

for $(r,s)\in\{-1\le r\le -1/2, r+s\ge 2\}\cup\{-1/2<r\le 1/2, r+3\le s\}\cup\{1/2<r, r+2\le s\}$ and $C=C(s,r,p,q,\rho)$. Solving the above inequality yields

$$\|U(t)\|_{H^r}+\|V(t)\|_{H^r}+\|F(t)\|_{H^r}+\|G(t)\|_{H^r}\le e^{CT}\left(\|U(0)\|_{H^r}+\|V(0)\|_{H^r}+\|F(0)\|_{H^r}+\|G(0)\|_{H^r}\right),$$

Noticing $F=f_1-f_2=\partial_x(u_1-u_2)=\partial_x U$ and $G=g_1-g_2=\partial_x(v_1-v_2)=\partial_x V$, by the above inequality we have

$$\|U(t)\|_{H^{r+1}}+\|V(t)\|_{H^{r+1}}\lesssim e^{CT}(\|U(0)\|_{H^{r+1}}+\|V(0)\|_{H^{r+1}}).$$

Lowering down the Sobolev index from $r+1$ to r and adjusting the range accordingly, we have

$$\|U(t)\|_{H^{r+1}}+\|V(t)\|_{H^{r+1}}\lesssim e^{CT}(\|U(0)\|_{H^{r+1}}+\|V(0)\|_{H^{r+1}}),$$

where $(r,s)\in A_1\doteq\{0\le r\le 1/2, r+s\ge 3\}\cup\{1/2<r\le 3/2, r+2\le s\}\cup\{3/2<r, r+1\le s\}$. Recalling the definition $z=(u_1,v_1)$ and $w=(u_2,v_2)$ reveals

$$\|z(t)-w(t)\|_{H^r}\lesssim e^{C_{r,s,p,q,\rho}T}\|z(0)-w(0)\|_{H^r},$$

where $(r,s)\in A_1\doteq\{0\le r\le 1/2, r+s\ge 3\}\cup\{1/2<r\le 3/2, r+2\le s\}\cup\{3/2<r, r+1\le s\}$. This completes the proof of Lipschitz continuity in the region A_1.

Hölder continuity in A_2. By the Lipschitz continuity in A_1 and the condition $r\le 3-s$, we obtain

$$\|z(t)-w(t)\|_{H^r}\le\|z(t)-w(t)\|_{H^{3-s}}\le e^{C_{r,s,p,q,\rho}T}\|z(0)-w(0)\|_{H^{3-s}}.$$

Interpolating between the H^r and the H^s norms (Lemma 4.3 with $\sigma_1=r$, $\sigma=s-3$ and $\sigma_2=s$) produces

$$\|z(0)-w(0)\|_{H^{3-s}}\le\|z(0)-w(0)\|_{H^r}^{\frac{2s-3}{s-r}}\|z(0)-w(0)\|_{H^s}^{\frac{3-s-r}{s-r}}\le c_{r,s,\rho}\|z(0)-w(0)\|_{H^r}^{\frac{2s-3}{s-r}}.$$

Hölder continuity in A_3. For the case $s-2\le r<s$, interpolating between the H^{s-2} and the H^s norms (Lemma 4.3 with $\sigma_1=s-2$, $\sigma=r$ and $\sigma_2=s$) generates

$$\|z(t)-w(t)\|_{H^r}\le\|z(t)-w(t)\|_{H^{s-2}}^{\frac{s-r}{2}}\|z(t)-w(t)\|_{H^s}^{\frac{r-s+2}{2}},$$

and by the well-posedness size estimate (4.1), we find

$$\|z(t)-w(t)\|_{H^s}\lesssim\|z_0\|_{H^s}+\|w_0\|_{H^s}\lesssim\rho,$$

therefore, we have

$$\|z(t)-w(t)\|_{H^r}\le C_{r,s,p,q,\rho}\|z(t)-w(t)\|_{H^{s-2}}^{\frac{s-r}{2}}.$$

By the Lipschitz continuity in A_1 and the condition $s-2\le r<s$, we obtain

$$\|z(t)-w(t)\|_{H^r}\le C_{r,s,\rho}\|z(0)-w(0)\|_{H^{s-2}}^{\frac{s-r}{2}}\le C_{r,s,\rho}\|z(0)-w(0)\|_{H^r}^{\frac{s-r}{2}},$$

which is the desired Hölder continuity in A_3.

Hölder continuity in A_4. For the case $s-1<r<s$, letting $\sigma_1=s-1$, $\sigma=r$ and $\sigma_2=s$ in Lemma 4.3 leads to

$$\|z(t)-w(t)\|_{H^r}\le\|z(t)-w(t)\|_{H^{s-1}}^{s-r}\|z(t)-w(t)\|_{H^s}^{r-s+1}.$$

By the Lipschitz continuity in A_1 for $r=s-1$ and the size estimate, we arrive at

$$\|z(t)-w(t)\|_{H^r}\le C_{r,s,\rho}\|z(0)-w(0)\|_{H^{s-1}}^{s-r}\le C_{r,s,\rho}\|z(0)-w(0)\|_{H^r}^{s-r},$$

which completes the proof of Theorem 1.3. □

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