On the interplay between Approximation Theory, Inverse Problems, and non-smooth solitons.

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The classical problem of a rotating rigid body

The classical Euler equation of a rotating rigid body:

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \boldsymbol{\omega}$$

where $\mathbf{M}, \boldsymbol{\omega} \in \mathbf{R}^3$ and $M_j = \sum_{k=1}^{3} I_{jk}\omega_k$ (I_{jk} - inertia tensor, symmetric positive definite)
In components

$$\frac{dM_i}{dt} = \sum_{j,k=1}^{3} \epsilon_{ijk} M_j \omega_k$$

ϵ_{ijk} is the Levi-Civita completely skew-symmetric tensor.

Lie algebra interpretation

ϵ_{ijk} defines the structure constants for the Lie algebra $so(3)$

Picture (Poincare-Arnold):

- Let G be a Lie group with the Lie algebra $\mathfrak{g}$. $\mathfrak{g}$ acts on itself via the *adjoint* representation ad.
- This action lifts to the dual $\mathfrak{g}^*$, and one gets the *co-adjoint* representation ad^* on the dual.
- Suppose $\mathfrak{g}$ is equipped with an inner product. The inner product induces an isomorphism $A : \mathfrak{g} \to \mathfrak{g}^*$

Then Euler's equation of the rigid body can be interpreted as

$$\boxed{\frac{dm}{dt} = ad^*_{A^{-1}m} m, \quad m \in \mathfrak{g}^*}$$

For the rigid body: $G = SO(3), A = I$(inertia tensor)
$\frac{d\mathbf{M}}{dt} = -I^{-1}\mathbf{M} \times \mathbf{M}$

Lie algebra of $\mathit{Diff}^+(S^1)$

The natural geometric way of interpreting elements $u \in \mathrm{diff}(S^1)$ is to view them as vector fields $u\,\partial_x$, and the dual space $\mathrm{diff}^*(S^1)$ as the space of quadratic differentials $\Omega^{\otimes^2}$, with the diffeomorphism-invariant pairing

$$\langle m\,dx^2, u\,\partial_x\rangle = \int_{S^1} mu\,dx.$$

The Lie bracket on $\mathfrak{g} = \mathrm{diff}(S^1)$ is the Lie bracket of vector fields,

$$[u\,\partial_x, v\,\partial_x] = (uv_x - u_x v)\,\partial_x,$$

and hence, if we integrate by parts,

$$\begin{aligned}\langle \mathrm{ad}^*_{u\,\partial_x}(m\,dx^2), v\,\partial_x\rangle &= -\langle m\,dx^2, [u\,\partial_x, v\,\partial_x]\rangle \\ &= -\int_{S^1} m\,(uv_x - u_x v)\,dx \\ &= \int_{S^1} \big((um)_x + u_x m\big)\,v\,dx,\end{aligned}$$

so

$$\boxed{\mathrm{ad}^*_{u\,\partial_x}(m\,dx^2) = \big((um)_x + u_x m\big)\,dx^2}$$

A priori there is of course no relation between m and u. However, if we equip the Lie algebra $\mathrm{diff}(S^1)$ with the H^1 inner product

$$(u\,\partial_x, v\,\partial_x) = \int_{S^1} \big(uv + u_x v_x\big)\, dx,$$

then after one integration by parts the inner product can be written

$$(u\,\partial_x, v\,\partial_x) = \int_{S^1} \big(u - u_{xx}\big)\, v\, dx = \langle Au\, dx^2, v\partial_x\rangle,$$

with $A = 1 - \partial_x^2$. In other words we have a map

$$\boxed{u \mapsto m = Au = u - u_{xx}}$$

Euler's equation on $Diff^+(S^1)$ (G. Misiolek 1998)

$$\boxed{m_t = (mu)_x + u_x m, \qquad \text{where } m = u - u_{xx}.}$$

This equation was proposed by Camassa and Holm in 1993 as a model of one-dimensional dispersive waves in shallow water.

Lax integrability

In $so(3)$ we have the equation $\mathbf{M_t} = [\omega, \mathbf{M}]$. One can construct a symmetric matrix J out of the tensor I so that the relation between $\mathbf{M}$ and ω takes the form

$$\mathbf{M} = \omega J + J\omega,$$

Then (Manakov) the Lax equation

$$\frac{d}{dt}(\mathbf{M} + zJ^2) = [\omega + zJ, \mathbf{M} + zJ^2]$$

is equivalent to the Euler's equation of the rigid body. This Lax equation can be viewed as a compatibility condition

$$\begin{aligned} (\mathbf{M} + zJ^2)\Psi &= \lambda\Psi \qquad \text{eigenvalue problem} \\ \Psi_t &= (\omega + zJ)\Psi \qquad \text{deformation} \end{aligned}$$

Lax integrability of the CH equation Consider

$$(-\partial_x^2 + \frac{1}{4})\psi = \frac{\lambda}{2} m\,\psi \qquad \text{eigenvalue problem}$$

$$\psi_t = \frac{1}{2}(\frac{1}{\lambda} + u_x)\psi - (\frac{1}{\lambda} + u)\psi \qquad \text{deformation equation}$$

The first miracle: the CH and Euler's equation of the rigid body are both Lax integrable.

The second miracle: peaked solitons (peakons)

CH admits weak solutions (with finite H^1-norm) in the form of peak-shaped travelling waves,

$$u(x,t) = c\,e^{-|x-ct|}, \qquad c \in \mathbb{R},$$

known as *peakons* (peaked solitons), on account of their obviously peaked shape together with the fact that they can also be combined via superposition to form *N-peakon* or *multipeakon* solutions of the form

$$u(x,t) = \sum_{k=1}^{N} m_k(t)\,e^{-|x-x_k(t)|},$$

or, since $m = (1-\partial_x^2)u$

$$\boxed{m(x,t) = 2\sum_{k=1}^{N} m_k(t)\,\delta\big(x - x_k(t)\big)}$$

Peakon equations

$$\dot{x}_k = u(x_k),$$
$$\dot{m}_k = -m_k \langle u_x \rangle (x_k),$$

$$1 \leq k \leq N$$

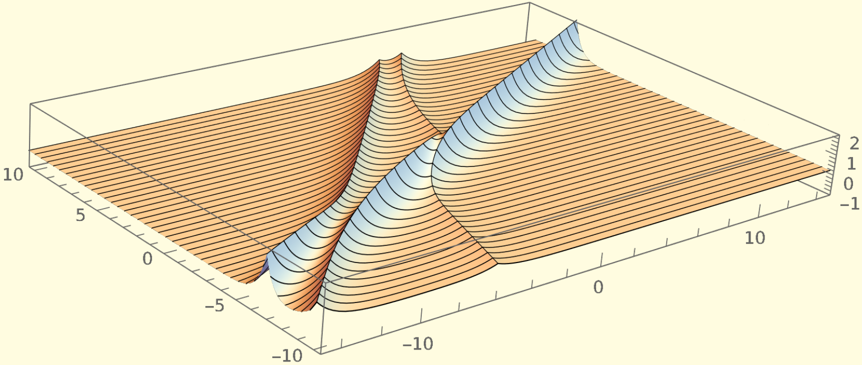


Figure: An example of a three-peakon solution of the Camassa–Holm equation . The graph of $u(x,t) = \sum_{k=1}^{3} m_k(t)\, e^{-|x-x_k(t)|}$ is plotted for $x \in [-15, 15]$ and $t \in [-10, 10]$ In this example, all amplitudes m_k are positive, so it is a *pure peakon solution* (i.e., there are no *antipeakons* with negative m_k).

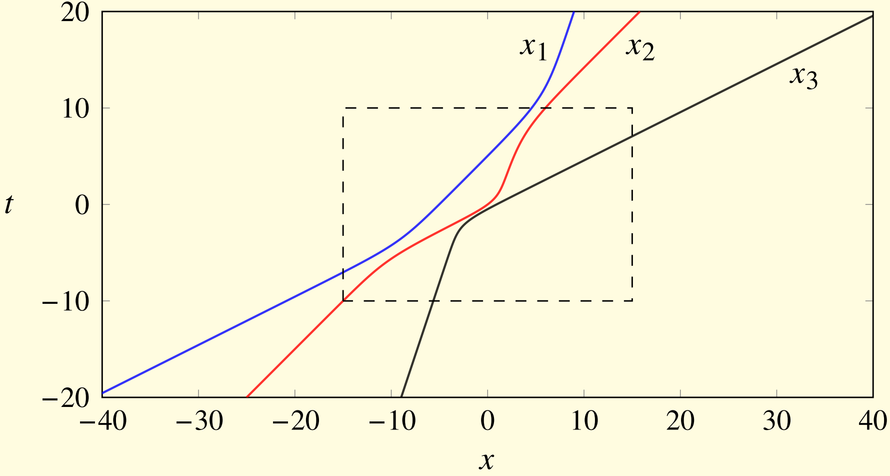


Figure: Positions $x = x_k(t)$ of the three individual peakons in the solution from Figure 1, with the dashed rectangle indicating the region shown there. Note that the ordering $x_1(t) < x_2(t) < x_3(t)$ is preserved for all t, and that the peakons asymptotically (as $t \to \pm\infty$) move in straight lines in the (x, t)-plane, like solitary travelling waves.

The string connection
R. Beals, D.Sattinger, J.S.

To start revealing that connection, (for now t is frozen) we make a Liouville transformation, i.e., a change of dependent and independent variables with the purpose of eliminating the constant term $-\frac{1}{4}$ in the differential operator $\partial_x^2 - \frac{1}{4}$ appearing in the first Lax equation.

$$\left(\partial_x^2 - \tfrac{1}{4}\right) \psi(x) = -\tfrac{1}{2}\lambda\, m(x)\, \psi(x), \qquad x \in \mathbb{R}.$$

Now let

$$y = \tanh(x/2), \qquad \psi(x) = \frac{\phi(y)}{\sqrt{1-y^2}}.$$

For smooth functions it is easily verified using the chain rule that the Liouville transformation turns the x-Lax operator into

$$-\partial_y^2 \phi(y) = \lambda\, g(y)\, \phi(y), \qquad -1 < y < 1,$$

where

$$\tfrac{1}{2}(1-y^2)^2 g(y) = m(x).$$

When $m = 2\sum_{k=1}^{N} m_k \delta\big(x - x_k)\big)$ we obtain the discrete measure g on the interval $(-1, 1)$, namely

$$g(y) = \sum_{k=1}^{N} g_k \, \delta(y - y_k), \qquad g_k = \frac{2m_k}{1 - y_k^2},$$

where (of course)

$$y_k = \tanh(x_k/2).$$

This situation corresponds to a *discrete string*:
an idealized object consisting of point masses of weight g_k at the positions y_k, connected by weightless string.

Next we define the so-called *Weyl function* of the discrete string:

$$W(\lambda) = \frac{\phi_y(1;\lambda)}{\phi(1;\lambda)}.$$

Clearly, this is a rational function with simple poles at the eigenvalues $\lambda_1, \ldots, \lambda_N$. It turns out to be somewhat more convenient to work with the modified Weyl function $\omega(\lambda) = W(\lambda)/\lambda$, so that $\omega(\lambda) = O(1/\lambda)$ as $\lambda \to \infty$. This modified Weyl function has an additional simple pole at $\lambda = \lambda_0 = 0$ with residue $W(0) = 1/2 = a_0$; denoting the residues at the other poles by a_k, the partial fractions decomposition of ω is

$$\omega(\lambda) = \frac{W(\lambda)}{\lambda} = \frac{1/2}{\lambda} + \sum_{k=1}^{N} \frac{a_k}{\lambda - \lambda_k} = \sum_{k=0}^{N} \frac{a_k}{\lambda - \lambda_k}.$$

The following Stieltjes continued fraction expansion holds:

$$\omega(\lambda) = \cfrac{1}{\lambda\, l_N + \cfrac{1}{-g_N + \cfrac{1}{\lambda\, l_{N-1} + \cfrac{1}{\ddots + \cfrac{1}{-g_1 + \cfrac{1}{\lambda\, l_0}}}}}}.$$

distances between the masses $l_j = y_{j+1} - y_j$

We can recover the coefficients of the continued fractions by studying approximations problems. A typical example (the diagonal Padé)

$$\boxed{Q_r(\lambda)\omega(\lambda) - P_r(\lambda) = O\left(\frac{1}{\lambda^{r+1}}\right)}$$

The polynomials $Q_r(\lambda)$ and $P_r(\lambda)$ (of degree r and $r-1$, respectively, and with $Q(0) = 1$) are uniquely determined by this condition. In fact Q_r determines P_r, and Q_r is computable using the moments of the measure α.

Let

$$\alpha_n = \int z^n \, d\alpha(z) = \sum_{k=0}^{N} \lambda_k^n \, a_k$$

be the nth moment of the spectral measure α. Then, using Cramer's rule, we obtain

$$Q_r(\lambda) = \frac{\begin{vmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^r \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_r \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{r-1} & \alpha_r & \alpha_{r+1} & \dots & \alpha_{2r-1} \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_r \\ \alpha_2 & \alpha_3 & \dots & \alpha_{r+1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_r & \alpha_{r+1} & \dots & \alpha_{2r-1} \end{vmatrix}}.$$

CH induced Isospectral Deformation

The CH equation induces an *isospectral deformation* of the string with Dirichlet boundary conditions; as time passes, the mass distribution of the string changes, but its Dirichlet spectrum remains the same. More precisely, if we split

$$\alpha = \frac{1}{2}\delta_0 + \sum_{k=1}^{N} a_k \delta_{\lambda_k}$$

and set

$$\hat{\alpha} := \sum_{k=1}^{N} a_k \delta_{\lambda_k}.$$

Then the CH flow on the string side reads

$$\boxed{\alpha(t) = \frac{1}{2}\delta_0 + e^{\frac{t}{\lambda}}\hat{\alpha}(0).}$$

Why is CH so special mathematically?

JS: CH is an isospectral deformation of an inhomogenous string. The connection to the string is, in my opinion, the crux of the matter!

Suppose we are given an arbitrary Hilbert space H (finite dimensional or infinite dimensional) and a self-adjoint operator A with positive, simple spectrum. Then A can be realized as a boundary value problem for an inhomogeneous string. This was proven by Krein around 1960.

If H is finite dimensional the corresponding string is a discrete string; we are in the peakon sector.

The Peakon Land (joint work with H. Lundmark)

By now (2022) we know a large number of peakon-bearing equations. The most popular other than CH are perhaps: the Degasperis-Processi equation

$$m_t + (um)_x + 2u_x m = 0, \qquad m = u - u_{xx}$$

and the V. Novikov equation

$$m_t + ((um)_x + 2u_x m)u = 0, \qquad m = u - u_{xx}.$$

The DP equation (after another Liouville transformation) is an isospectral deformation of the *cubic string*

$$-\partial_y^3\varphi(y) = \lambda g(y)\varphi(y), \qquad \varphi(-1) = \varphi_y(-1) = 0 = \varphi(1).$$

This is a **non-selfadjoint problem***, but for positive* m*, and thus* g*, the spectrum is positive and simple!!!*

NV (with H. Lundmark and A. Hone)

NV is an isospectral deformation of the *dual cubic string (after a Liouville transformation)*

$$\frac{\partial}{\partial y}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix} = \begin{bmatrix}0 & g(y) & 0\\ 0 & 0 & g(y)\\ -\lambda & 0 & 0\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix}, \qquad \varphi_2(-1) = \varphi_3(-1) = 0 = \phi_3(1).$$

Again, this is a non-selfadjoint problem, but for positive g the spectrum is positive and simple.

Dual strings

Roughly, if the original discrete string (or discrete cubic string) is characterized by distances $\{l_j\}$ between the masses, and the masses $\{g_j\}$, then for the dual string (dual cubic string) the new distances are given by $\{g_j\}$ and the new masses are $\{l_j\}$. In other words

$$l_j \leftrightarrow g_j$$

In this sense the DP and NV are in **duality** for positive measures (for peakons, no mixed peakons-antipeakons).

Approximation problems

For the cubic string there are **two** Weyl functions $W(\lambda)$ and $Z(\lambda)$, and solving the inverse problem for the cubic string amounts to solving the following **Hermite-Padé** approximation:

$$Q(\lambda)W(\lambda) - P(\lambda) = O(1), \quad Q(\lambda)Z(\lambda) - \widehat{P}(\lambda) = O(1)$$
$$Q(\lambda)Z(-\lambda) - P(\lambda)W(-\lambda) + \widehat{P}(\lambda) = O(\lambda^{-r}),$$
$$\deg Q(\lambda) = \deg \widehat{P}(\lambda) = r - 1,$$
$$P(0) = 1, \widehat{P}(0) = 0.$$

Cauchy biorthogonal polynomials (with M. Bertola and M. Gekhtman)

The solution to these approximation problems can be written in terms of an interesting class of biorthogonal polynomials (Cauchy biorthogonal polynomials)

Definition

Let α and β be two positive measures with support inside $\mathbf{R}_+$. Then the family of biorthogonal polynomials $\{q_n(x), p_n(x), n \in \mathbf{N}\}$ satisfies

$$\langle q_m, p_n \rangle = \int_{\mathbf{R}_+^2} \frac{q_m(x)p_n(y)}{x+y} d\alpha(x) d\beta(y) = \delta_{mn}$$

DP: $\alpha(x) = \delta(x) + \sum_{k=1}^{N} a_k \delta(x - \lambda_k), \quad \beta(x) = x\alpha(x),$
NV: $\alpha(x) = \sum_{k=1}^{N} a_k \delta(x - \lambda_k), \quad \beta(x) = \alpha(x).$

The spectra (in the pure peakon cases) is positive and simple

The cubic string and the dual cubic string are not self-adjoint but still have positive simple spectra. Why?

The cubic string and the dual cubic string are non-selfadjoint oscillatory systems in the sense of Gantmakher and Krein

NV2; joint work with X. Chang

We consider the system (introduced by Hongmin Li in 2019)

$$
\begin{aligned}
&m_t + (uvm)_x + u_x vm = 0,\\
&n_t + (uvn)_x + uv_x n = 0,\\
&m = u - u_{xx}, \qquad n = v - v_{xx}.
\end{aligned}
$$

Then

$$
u = \sum_{j=1}^{N} m_j e^{-|x-x_j|}, \qquad v = \sum_{j=1}^{N} n_j e^{-|x-x_j|},
$$

and the equations of motion for peakons read:

$$
\boxed{
\begin{gathered}
\dot{x}_j = u(x_j)v(x_j),\\
\dot{m}_j = -m_j\langle u_x\rangle(x_j)v(x_j), \qquad \dot{n}_j = -n_j\langle v_x\rangle(x_j)u(x_j).
\end{gathered}
}
$$

BVP for NV2

We consider, following Hongmin Li,

$$D_x \begin{bmatrix} \psi_1 \\ \psi_{\mathbf{2}} \\ \psi_3 \end{bmatrix} = \begin{bmatrix} 0 & z\mathbf{m}^* & 1 \\ 0_{2\times 1} & 0_{2\times 2} & z\mathbf{m} \\ 1 & 0_{1\times 2} & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_{\mathbf{2}} \\ \psi_3 \end{bmatrix}$$

where ψ_1, ψ_3 are scalar quantities $(\in M_{1,1})$, $\psi_{\mathbf{2}} \in M_{2,1}$,

$$\mathbf{m}^* = \begin{bmatrix} n & m \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} m \\ n \end{bmatrix},$$

and $z \in \mathbb{C}$ is a spectral parameter.
The admissible boundary conditions are :

$$\psi_3(-\infty) = 0,\ \psi_{\mathbf{2}}(-\infty) = 0_{2\times 1},\ \text{and } \psi_3(+\infty) = 0.$$

Peakon Sector

When the measures are finite and discrete we will write

$$\mathbf{m}^* = 2\sum_{j=1}^{N} \mathbf{m}_j^* \delta_{x_j}, \qquad \mathbf{m} = 2\sum_{j=1}^{N} \mathbf{m}_j \delta_{x_j},$$

$$\mathbf{m}_j^* = [n_j m_j], \qquad \mathbf{m}_j = \begin{bmatrix} m_j \\ n_j \end{bmatrix}.$$

Spectrum

First step: **Set**

$$l = -z^2.$$

For the case of the finite discrete measure given we obtain a matrix eigenvalue problem for the components of $\langle \psi_{\mathbf{2}} \rangle$. Let us define

$$\Psi = \begin{bmatrix} \langle \psi_{\mathbf{2}} \rangle (x_1) \\ \langle \psi_{\mathbf{2}} \rangle (x_2) \\ \vdots \\ \langle \psi_{\mathbf{2}} \rangle (x_N) \end{bmatrix} \in M_{2N,1},$$

$$P = \begin{bmatrix} \mathbf{m}_1 & 0_{2\times 1} & 0_{2\times 1} & \dots & 0_{2\times 1} \\ 0_{2\times 1} & \mathbf{m}_2 & 0_{2\times 1} & \dots & 0_{2\times 1} \\ \vdots & \dots & \dots & \dots & 0_{2\times 1} \\ 0_{2\times 1} & 0_{2\times 1} & \dots & \dots & \mathbf{m}_N \end{bmatrix} \in M_{2N,N}$$

$$E = \begin{bmatrix} 1 & e^{-|x_1-x_2|} & \dots & e^{-|x_1-x_N|} \\ e^{-|x_2-x_1|} & 1 & \dots & e^{-|x_2-x_N|} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-|x_N-x_1|} & e^{-|x_N-x_2|} & \dots & 1 \end{bmatrix} \in M_{N,N},$$

$$T = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & 1 & 0 \\ 2 & 2 & 2 & \dots & 1 \end{bmatrix} \in M_{N,N},$$

Proposition

The column matrix $\Psi \in M_{2N,1}$ *solves the eigenvalue problem*

$$\Psi = \lambda \Big[(T \otimes 1_2) P E P^* \Big] \Psi .$$

Proposition

1. *the spectrum of the original boundary value problem is given by the zeros of* $A(\lambda)$ *(with the caveat that* $\lambda = -z^2$*).*
2. $A(\lambda)$ *is time invariant.*

The matrix $\Big[(T \otimes 1_2) P^* E P \Big] \in M_{2N,2N}$ generalizes the matrix $TPEP \in M_{N,N}$ where $P = \mathrm{diag}(m_1, m_2, \dots, m_N) \in M_{N,N}$, occurring in the treatment of the peakon problem for the NV equation by Hone, Lundmark and JS.

The matrix $(T \otimes 1_2)PEP^*$ is not oscillatory (unlike in the NV case). It is very hard to determine the spectrum.

Since the problem is isospectral perhaps we can evolve $(T \otimes 1_2)PEP^\sigma$ to a time t_0 at which it becomes simple?

1. First idea: look at

$$E = \begin{bmatrix} 1 & e^{-|x_1-x_2|} & \dots & e^{-|x_1-x_N|} \\ e^{-|x_2-x_1|} & 1 & \dots & e^{-|x_2-x_N|} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-|x_N-x_1|} & e^{-|x_N-x_2|} & \dots & 1 \end{bmatrix} \in M_{N,N},$$

2. Perhaps we can " kill off" all those exponentials? Then E becomes the identity matrix.

Can we prove that $|x_i - x_j| \to \infty$ if $i \neq j$ at some time t_0?

3. NV2 peakons scatter, i.e. $|x_i - x_j| \to \infty$ if $i \neq j$ and $t \to \infty$.
4. you need to prove global existence of peakon flows (for postive measures)
5. if peakons scatter then asymptotically they are free particles, i.e.

$$x_j(t) = v_j t + O(1),$$

and $v_1 < v_2 < \cdots < v_N$

6. Asymptotically,

$$\boxed{x_j(t) = m_j(\infty)n_j(\infty)t + O(1), \qquad t \to \infty}$$

7. The eigenvalues of the eigenvalue problem are:

$$\boxed{\lambda_j = \frac{1}{2m_j(\infty)n_j(\infty)}, \qquad 1 \le j \le N.}$$

In particular, all eigenvalues are positive and simple.

A beam problem; joint work with R. Beals

$$\boxed{D_x^2[rD_x^2\phi] = \lambda^2\rho\phi, \qquad -1 < x < 1}$$

Lemma (R. Beals and J.S)

Set $\eta = 1/r$. Then the beam problem is equivalent to

$$D_x^2\Phi = \lambda\mathcal{M}\Phi, \qquad \mathcal{M} = \begin{bmatrix} 0 & \eta \\ \rho & 0 \end{bmatrix}, -1 < x < 1$$

The Euler beam is a “string” with an internal structure. (matrix string)

Since the space of initial conditions is 4 dimensional, putting boundary conditions amounts to choosing lower dimensional subspaces of $\mathbb{R}^4$.

Definition (Dirichlet BC)

Let Φ be a 2×2 solution to the matrix string equation such that $\Phi(-1, \lambda) = 0, \Phi_x(-1, \lambda) = \mathbf{1}$. Then the Dirichlet spectrum $\mathcal{S}_{\mathcal{M}} = \{\lambda \in \mathbb{C} : \det \Phi(1, \lambda) = 0\}$

Isospectral deformations of the DD beam

$$\boxed{\partial_t \Phi = (a + b\partial_x)\Phi}$$

Again, only deformations regular at $\lambda = \infty$ work for measures. The simplest (level 1, only $1/\lambda$ power)

Deformation equations

Recall that $\Phi(x,\lambda)$ satisfies $D_x^2\Phi = \lambda\mathcal{M}\Phi$ where $\Phi(-1,\lambda)=0, \Phi'(-1,\lambda)=\mathbf{1}$ and $\mathcal{M} = \begin{bmatrix} 0 & \eta \\ \rho & 0 \end{bmatrix}$.

Then the isospectral, level one, evolution equations for the DD beam are

$$\boxed{\eta_t = (\alpha\eta)_x + \alpha_x\eta + \beta\eta, \quad \rho_t = (\alpha\rho)_x + \alpha_x\rho - \beta\rho}$$

It is instructive to see how these equations look if the interval $[-1,1]$ is mapped to $\mathbb{R}$:

$\rho \to m, \quad \eta \to n, \quad \alpha \to u, \quad \beta \to v, \quad \mathcal{M} \to M = \begin{bmatrix} 0 & n \\ m & 0 \end{bmatrix}$

- $D_x^2 \Phi = \big(\mathbf{1} + \lambda M\big)\Phi$
- $n_t = (un)_x + u_x n + vn, \quad m_t = (um)_x + u_x m - vm$
- $v_x = (m - n), \quad u - u_{xx} = m + n$
- Peakon equations:

$$\begin{aligned} \dot{x}_i &= u(x_i), \\ \dot{m}_j &= -m_j \big\langle u_x - v \big\rangle (x_i) \\ \dot{n}_j &= -n_j \big\langle u_x + v \big\rangle (x_i) \end{aligned}$$

Stieltjes and strings

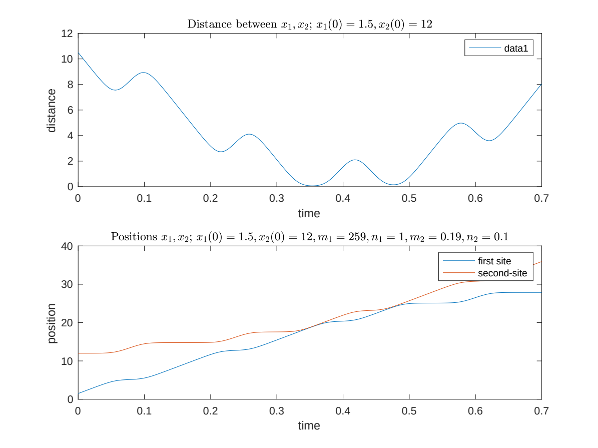
For the string, the resolvent (the modified Weyl function)

$$\boxed{\omega(z) = \cfrac{1}{l_d z + \cfrac{1}{m_d + \cfrac{1}{\ddots + \frac{1}{l_0 z}}}}}$$

Stieltjes and beams

For the beam, the modified Weyl function $\omega(\lambda) = \frac{1}{\lambda}\Phi_x(1,\lambda)\Phi(1,\lambda)^{-1}$ (a 2×2 matrix)

$$\boxed{\omega(\lambda) = \cfrac{1}{l_d\mathbf{1}\lambda + \cfrac{1}{\mathcal{M}_d + \cfrac{1}{\ddots + \frac{1}{l_0\mathbf{1}\lambda}}}}}$$


Distance between x_1, x_2; $x_1(0) = 1.5, x_2(0) = 12$
data1
distance
time
Positions x_1, x_2; $x_1(0) = 1.5, x_2(0) = 12, m_1 = 259, n_1 = 1, m_2 = 0.19, n_2 = 0.1$
first site
second-site
position
time

Thank you !