Two nonlocal problems: Equations in infinitely many derivatives and the Cauchy problem for the Kadomtsev-Petviashvili hierarchy

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Abstract

First I'll talk briefly on "nonlocal equations" and then I'll focus on the main subject of this presentation, the Cauchy problem of the Kadomtsev-Petviashvili (KP) hierarchy.

My main goal is to show that one can solve *all* the equations of the KP hierarchy using a factorization of an infinite dimensional Lie group of pseudo-differential operators. This result can be studied in several contexts:

- *Algebraic*: formal pseudo-differential operators are defined on algebras equipped with derivations and valuations, and the group is a formal object.

- *Geometric*: formal pseudo-differential operators are defined on algebras equipped with a Frölicher (or Fréchet) structure, and the group is a Frölicher Lie group.

- *Analytic*: pseudo-differential operators are not formal. The KP hierarchy is understood as a non-linear equation on a Frölicher group built with the help of a class of true pseudo-differential operators.

1. Nonlocal equations

In the paper

"A new formulation of the initial value problem for nonlocal theories", *Nuclear Physics B* 845 (2011), 1–29,

Neil Barnaby considers the nonlocal equation

$$e^{-2\Box}(\Box + 1)\phi = \alpha\phi^2 \ . \tag{1}$$

Let me take this example as motivation for the analytic study of nonlocal equations

$$f(\Delta)\phi = U(\cdot, \phi) \ . \tag{2}$$

So far the following has been achieved:

1. A good definition of classes $\mathcal{G}^\beta$ of "symbols" f.
2. A good definition of domains $\mathcal{H}^\beta(f)$ for the operator $f(\Delta)$.
3. Several existence theorems for regular solutions to Equation (2).

Here is a typical result:

Theorem 1. *Let us assume that* $\alpha > 1$ *is a constant, that* $f \in \mathcal{G}^\beta$ *for* $\beta > \frac{n}{2}\left(\frac{\alpha-1}{\alpha}\right)$, *and that* U *is spherically symmetric with respect to* x. *Assume also that there exist functions* $h \in L^2(\mathbb{R}^n)$, $g \in L^{\frac{2\alpha}{\alpha-1}}(\mathbb{R}^n)$ *such that the following two inequalities hold:*

$$|U(x,y)+y| \leq C(|h(x)| + |y|^\alpha), \quad |\frac{\partial}{\partial y}(U(x,y)+y)| \leq C(|g(x)| + |y|^{\alpha-1})$$

for some constant $C > 0$. *Then, there exist* $0 < \epsilon < 1$ *and* $0 < \rho_\epsilon < 1$ *such that, whenever* $\|h\|_{L^2(\mathbb{R}^n)} < \rho_\epsilon$, *there is a radial solution* $u \in \mathcal{H}^\beta(f)$ *to the equation*

$$f(\Delta)u - U(\cdot, u) = 0$$

with $\|u\|_{L_r^{2\alpha}(\mathbb{R}^n)} \leq \epsilon$.

Let me be specific about the class $\mathcal{G}^\beta$: it is the set of measurable functions f satisfying

(P) The function $s \mapsto f(-s^2)$ is non-negative.

(E_β) There exist real numbers $\beta, R, M > 0$ such that

$$M(1 + |\xi|^2)^{\frac{\beta}{2}} \leq f\left(-|\xi|^2\right) \text{ for all } \xi \text{ with } |\xi| > R\,.$$

If one fixes $f \in \mathcal{G}^\beta$, $\mathcal{H}^\beta(f)$ is the set of all real-valued functions g on $\mathbb{R}^n$ such that g is measurable, the Fourier transform $\mathcal{F}(g)$ exists, and

$$\int_{\mathbb{R}^n} \left[1 + f(-|\xi|^2)\right]^2 |\mathcal{F}(g)(\xi)|^2\, d\xi < \infty\,.$$

It follows from the definition of $\mathcal{H}^\beta(f)$ that $\mathcal{H}^\beta(f) \subseteq L^2(\mathbb{R}^n)$.

The proof of Theorem 1 relies on a fixed point argument. It uses essentially theorems on compact embeddings of Sobolev spaces of functions possessing spherical symmetry.

More sophisticated results can be proven: Equation (1) can be studied in the context of general L^p-spaces.

This theory is summarized in the papers

1. Przemyslaw Górka, Humberto Prado, Enrique G. Reyes "On a General Class of Nonlocal Equations". *Annales Henri Poincaré*, May 2013, Volume 14, Issue 4, pp 947-966.

2. Mauricio Bravo, Humberto Prado, Enrique G. Reyes, "Nonlinear pseudo-differential equations defined by elliptic symbols on $L^p(\mathbb{R}^n)$ and the fractional Laplacian". *Israel Journal of Mathematics*, May 2019, Volume 231, Issue 1, pp 269-301.

Let me go back to Barnaby's paper. His example of nonlocal equation was

$$e^{-2\Box}(\Box + 1)\phi = \alpha\phi^2 \ , \tag{3}$$

in which α is a constant. In de Sitter space-time we have $\Box = -\partial_t^2 - \beta\partial_t$ for a constant β, and Equation (3) becomes

$$e^{2(\partial_t^2+\beta\partial_t)}(\partial_t^2 + \beta\partial_t - 1)\phi = -\alpha\phi^2 \ .$$

I take this as motivation for the study of ordinary nonlocal equations

$$f(\partial_t)\phi(t) = J(t) \ .$$

Now, trying to understand these equations I came across the papers

1. N. Barnaby and N. Kamran, Dynamics with infinitely many derivatives: the initial value problem. *J. High Energy Physics* 2008 no. 02, Paper 008, 40 pp.

2. N. Barnaby and N. Kamran, Dynamics with infinitely many derivatives: variable coefficient equations. *J. High Energy Physics* 2008 no. 12, Paper 022, 27 pp.

These works led me to study Laplace transform!

The main technical problem to be solved if one wishes to understand the equation

$$f(\partial_t)\phi(t) = J(t)\ .$$

is to find a reasonable definition and domain for general operators of the form $f(\partial_t)$. A first try was discussed in

P Gorka, H Prado and E G Reyes, "The initial value problem for ordinary differential equations with infinitely many derivatives". *Classical and Quantum Gravity* 29 (2012) 065017 (15pp).

A better approach is in

Alan Chavez, Humberto Prado, Enrique G. Reyes, "A Laplace transform approach to linear equations with infinitely many derivatives and zeta-nonlocal field equations". *Advances in Theoretical and Mathematical Physics* 23 (2019), 1771–1804. (published online: May 15, 2020).

Let me mention the main tools used in the later paper:

1. The pth Lebesgue space $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$, and the qth Hardy space $H^q(\mathbb{C}_+)$.

2. The classic Representation theorem presented by G. Doetsch in

   "Bedingungen für die Darstellbarkeit einer Funktion als Laplace-integral und eine Umkehrformel für die Laplace-Transformation". *Math. Z.* 42 (1937), no. 1, 263–286.

Theorem 2.

(i) If $\phi \in L^p(\mathbb{R}_+)$, where $1 < p \leq 2$, and $\Phi =$ Laplace transform of ϕ, then $\Phi \in H^{p'}(\mathbb{C}_+)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

(ii) If $\Phi \in H^p(\mathbb{C}_+)$, where $1 < p \leq 2$, then there exists $\phi \in L^{p'}(0, \infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ such that $\Phi = \mathcal{L}(\phi)$. The function ϕ is given by the inversion formula

$$\phi(t) := \lim_{v \to \infty} \frac{1}{2\pi} \int_{-v}^{v} e^{(\sigma + i\eta)t} \Phi(\sigma + i\eta) d\eta \ , \qquad \sigma \geq 0 \ ,$$

in which the limit is understood in $L^{p'}(\mathbb{R}_+)$.

The Doetsch correspondence allows us to define $f(\partial_t)$ as an operator from $L^p(\mathbb{R}_+)$ spaces to $H^q(\mathbb{C}_+)$ spaces and (for instance) to study the initial value problem for the equation

$$\zeta(\partial_t + h)\phi = J(t) \qquad t \geq 0$$

and also for

$$\zeta(\partial_t^2 + h)\phi = J(t) \ , \qquad t \geq 0 \ ,$$

in which ζ is the Riemann zeta function, although a full analysis of the second equation requires a generalization of the theory. (Alan Chavez, *Universidad de Chile Ph.D. Thesis*, 2018 & *The Borel transform and linear nonlocal equations: applications to zeta-nonlocal field models*. With Alan Chávez and Humberto Prado. Submitted, 2019).

2. The Kadomtsev-Petviashvili hierarchy

My original question:

Given the equation $u_t = F(u, u_x, u_{xx}, \cdots)$, understand Peter Olver's equation

$$R_t = [F_*, R] \ , \tag{4}$$

in which F_* is the formal linearization of F,

$$F_* = \sum \frac{\partial F}{\partial u_{x \cdots x}} D_x^k \ ,$$

D_x is total derivative with respect to x, and R is a formal pseudodifferential operator,

$$R = \sum_{-\infty < k \leq N} a_k D_x^k$$

whose coefficients are smooth functions on some jet bundles of E.

Now, trying to understand (4), I learnt that one of the amazing observations about the KdV equation

$$4u_t = u_{xxx} + 6uu_x$$

is that it can be reformulated as the Lax equation

$$L_t = [P, L]$$

with

$$L = \left(\frac{\partial}{\partial x}\right)^2 + u\,, \qquad P = \left(\frac{\partial}{\partial x}\right)^3 + \frac{3}{2}u\,\frac{\partial}{\partial x} + \frac{3}{4}u_x\,.$$

The Lax equation is similar to (4)!

So, instead of trying to understand (4) I decided to study Lax equations.

Let L be the formal pseudo-differential operator

$$L = \frac{\partial}{\partial x} + u_1 \left(\frac{\partial}{\partial x} \right)^{-1} + u_2 \left(\frac{\partial}{\partial x} \right)^{-2} + \dots \; ,$$

in which $u_1, u_2, \dots,$ are an infinite number of variables —to be identified with dependent variables of systems of partial differential equations— and consider the differential operator

$$(L^m)_+ \; ,$$

in which $(\cdot)_+$ indicates projection into the space of differential operators.

The Kadomtsev-Petviashvili (KP) hierarchy is the infinite system of equations

$$\partial_{t_m} L = [(L^m)_+, L] \; , \qquad m = 1, 2, 3, \cdots . \tag{5}$$

If $m = 1$, Equations (5) reduce to the infinite system of equations

$$\frac{\partial}{\partial t_1} u_m = \frac{\partial}{\partial x} u_m \, , \quad m = 1, 2, 3, \ldots \, ,$$

and therefore we usually identify t_1 with the independent variable x.

For a fixed value of m, Equations (5) encode an infinite system of nonlinear partial differential equations for dependent variables u_k, $k = 1, 2, 3, \ldots$, in two independent variables, t_m and x.

Lemma 1. *If L satisfies*

$$\partial_{t_m} L = [L_+^m, L] \, , \quad m = 1, 2, 3, \ldots \, ,$$

then

$$\partial_{t_r} L^s - \partial_{t_s} L^r = [L^r, L^s] \, .$$

These systems of equations are generalizations of KdV!

Example: If $s = 3$ and $r = 2$, the equation

$$\partial_{t_r} L^s - \partial_{t_s} L^r = [L^r, L^s]$$

can be written as

$$3u_{yy} = (4u_t - u_{xxx} - 6uu_x)_x$$

for $2u_1 = u$, $t_2 = y$ and $t_3 = t$.

This is the *Kadomtsev-Petviashvili equation*.

I take the previous discussion as motivation for the study of the Kadomtsev-Petviashvili hierarchy of equations in a PDE sense: I wish to understand its Cauchy problem.

I will show that the following plan can be carried out:

1. Define pseudo-differential operators in an algebraic context. Use them to define an infinite-dimensional Lie algebra $\Psi(A)$.
2. Construct a group $G(\Psi)$ with Lie algebra $\Psi(A)$.
3. Present a factorization result for the group $G(\Psi)$.
4. Use the factorization to solve the Cauchy problem for the KP equations.

Let me make a remark: working on this problem one learns the following:

The theory of infinite-dimensional Lie algebras is (well!) algebraic...

The theory of infinite dimensional Lie groups has a strong analytic flavour.

Key references for these topics are

1. Victor Kac, *Infinite dimensional Lie algebras*, 3rd edition, Cambridge University Press (1990).

2. Laurent Guieu, Claude Roger, *L'algèbre et le groupe de Virasoro: aspects géométriques et algébriques, généralisations*, Université de Montréal, 2007.

A typical Lie group question of importance for this talk is:

Given an infinite dimensional Lie group G with Lie algebra $Lie(G)$,

Is there a well-defined exponential map from $Lie(G)$ to G?

Answer: No! If $G = Diff(S^1)$, then $Lie(G) = Vect(S^1)$ and the exponential map is not a local diffeomorphism.

An excellent reference is:

Boris Khesin and Robert Wendt,

The Geometry of Infinite-Dimensional Groups, Springer, 2009.

3. A preliminary calculation

Let us consider the *Toda lattice*

$$\frac{dL}{dt} + [L, B] = 0 \ ,$$

in which

$$L = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 & 0 \\ b_0 & a_1 & b_1 & 0 & \cdots & 0 \\ 0 & b_1 & a_2 & b_2 & \cdots & 0 \\ \vdots & & & & & \vdots \\ & & b_{N-4} & a_{N-3} & b_{N-3} & 0 \\ & & 0 & b_{N-3} & a_{N-2} & b_{N-2} \\ & & 0 & 0 & b_{N-2} & a_{N-1} \end{pmatrix}$$

and

$B =$ upper triangular part of $L -$ lower triangular part of L,

$$B = \begin{pmatrix} 0 & b_0 & 0 & \cdots & 0 & 0 \\ -b_0 & 0 & b_1 & 0 & \cdots & 0 \\ 0 & -b_1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ & & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & b_{N-2} \\ & & 0 & 0 & -b_{N-2} & 0 \end{pmatrix}$$

We use the following fact:

Every invertible matrix $A \in SL(n, \mathbb{R})$ can be written as a product,

$$A = Q\, R\,,$$

in which $Q \in SO(n, \mathbb{R})$ is an orthogonal matrix and R is an upper triangular matrix.

Theorem 3. *Given an initial condition L_0, we define matrices $Q(t)$ and $R(t)$ via*

$$\exp(t\,L_0) = Q(t)\,R(t)\ .$$

The solution to the initial value problem

$$\frac{dL}{dt} + [L, B] = 0\ , \qquad L(0) = L_0\ ,$$

is

$$L(t) = Q(t)^{-1}\,L_0\,Q(t)\ .$$

Reference: W.W. Symes, "Hamiltonian group actions and integrable systems", *Physica* 1 D (1980), 339-374.

4. Formal pseudo-differential operators and infinite dimensional Lie groups

We begin with the construction of formal pseudo-differential operators and the Lie algebra $\Psi(A)$.

Definition of the Lie algebra $\Psi(A)$.

We let A be an associative and commutative K–algebra with unit 1, in which K is an arbitrary field of characteristic zero. We assume that A is equipped with a derivation, that is, with a K–linear map

$$D : A \to A$$

satisfying the Leibniz rule

$$D(f \cdot g) = (Df) \cdot g + f \cdot (Dg)$$

for all $f, g \in A$.

Let ξ be a formal variable, not in A. The *algebra of symbols* over A is the vector space

$$\Psi_\xi(A) = \left\{ P_\xi = \sum_{\nu \in \mathbf{Z}} a_\nu \, \xi^\nu : a_\nu \in A \, , \; a_\nu = 0 \text{ for } \nu \gg 0 \right\}$$

equipped with the associative multiplication

$$P_\xi \circ Q_\xi = \sum_{k \geq 0} \frac{1}{k!} \frac{\partial^k P_\xi}{\partial \xi^k} D^k Q_\xi \, .$$

Multiplication on the right hand side is standard multiplication of Laurent series in ξ with coefficients in A.

A is included in $\Psi_\xi(A)$, and for $f \in A$ we have

$$\xi^{-1} \circ f = f\xi^{-1} - D(f)\xi^{-2} + D^2(f)\xi^{-3} - \cdots$$

and more generally, for $\nu \in \mathbb{Z}$,

$$\xi^\nu \circ f = \sum_{k \geq 0} \frac{\nu(\nu-1)\ldots(\nu-k+1)}{k!} \left(D^k f\right) \xi^{\nu-k} ,$$

so that the multiplication $\circ$ mirrors the extension of the Leibniz rule to negative powers of the derivative D.

The algebra of formal pseudo-differential operators over A is the set

$$\Psi(A) = \left\{ P = \sum_{\nu \in \mathbf{Z}} a_\nu D^\nu : a_\nu \in A\,,\ a_\nu = 0 \text{ for } \nu \gg 0 \right\} ,$$

in which a multiplication $\circ$ is defined so that the map

$$\sum_{\nu \in \mathbf{Z}} a_\nu \xi^\nu \mapsto \sum_{\nu \in \mathbf{Z}} a_\nu D^\nu$$

from $\Psi_\xi(A)$ to $\Psi(A)$ is an algebra homomorphism.

The algebra $\Psi(A)$ is associative but not commutative.

It becomes a Lie algebra over K if we define

$$[P, Q] = P \circ Q - Q \circ P\ .$$

Construction of a group $G(\Psi(A))$ with Lie algebra $\Psi(A)$.

The Lie algebra $\Psi(A)$ admits a left A–module direct sum decomposition

$$\Psi(A) = \mathcal{D}_A \oplus \mathcal{I}_A \, , \tag{6}$$

in which $\mathcal{D}_A$ is the Lie subalgebra of all differential operators of order greater or equal to zero, and $\mathcal{I}_A$ is the Lie subalgebra of integral operators, that is, the set of all formal pseudo-differential operators in $\Psi(A)$ of order at most -1.

Goal: To construct formal Lie groups $G(\Psi(A))$, $G_+(\mathcal{D}_A)$, and $G_-(\mathcal{I}_A)$ with Lie algebras $\Psi(A)$, $\mathcal{D}_A$, and $\mathcal{I}_A$ respectively, and such that

$$G(\Psi(A)) = G_+(\mathcal{D}_A) \cdot G_-(\mathcal{I}_A) \, .$$

I use Motohico Mulase's beautiful paper

Mulase, M.; Solvability of the super KP equation and a generalization of the Birkhoff decomposition. *Invent. Math.* **92** (1988), 1–46.

I need to take a special algebra A:

Let R be a fixed commutative algebra over a field K equipped with a derivation D.

A is the ring of formal power series over R in an infinite number of variables $\tau_1, \tau_2, \cdots$.

Construction of A:

We consider a countable set of indices I, and we take T as the additive monoid of all sequences of natural numbers $t = (n_i)_{i \in I}$ such that $n_i = 0$ except for a finite number of indices.

A *formal power series* $u \in A$ is a function from T to R,

$$u = (u_t)_{t \in T}$$

.

Consider an infinite number of formal variables τ_i, $i \in I$ and set $\tau = (\tau_1, \tau_2, \cdots)$. Then, the formal power series u is also written as

$$u = \sum_{t \in T} u_t \tau^t \; .$$

The set of formal power series A is a commutative algebra with unit, and the derivation D on R extends to a derivation on A via

$$Du = \sum_{t \in T} (Du_t)\tau^t.$$

Let $u = \sum_{t \in T} u_t \tau^t \in A$, $u \neq 0$. If $t = (n_i)_{i \in I}$, we set $\mid t \mid = \sum n_i$.

The terms $u_t \tau^t$ such that $\mid t \mid = p$ are called *terms of total degree* p.

The formal power series u_p is the series whose terms of total degree p are those of u, and whose other terms are zero.

For a formal series $u \neq 0$, the least integer $p \geqslant 0$ such that $u_p \neq 0$ is called the *order* of u, and it is denoted by $ord_t(u)$. We extend this definition to the case $u = 0$ setting $ord_t(0) = \infty$.

Lemma 2. *The following properties hold: if u, v are formal power series different from zero, then*

$$ord_t(u+v) \geqslant inf(ord_t(u), ord_t(v)) \ , \qquad if\ u+v \neq 0,$$

$$ord_t(u+v) = inf(ord_t(u), ord_t(v)) \ , \qquad if\ ord_t(u) \neq ord_t(v),$$

$$ord_t(uv) \geqslant ord_t(u) + ord_t(v) \ , \qquad if\ uv \neq 0.$$

Following Motohico Mulase (*Inventiones Mathematicae* 1988), we define the algebra

$$\widehat{\Psi}(A) \;=\; \Big\{ P = \sum_{l \in \mathbb{Z}} a_l \, D^l : a_l \in A \text{ and } \exists C \in \mathbb{R}^+, N \in \mathbb{Z}^+ \\ \text{so that } \; ord_t(a_l) > C\, l - N \;\; \forall\, l \gg 0 \}$$

and the subalgebra

$$\widehat{\mathcal{D}}_A = \left\{ P = \sum_{l \in \mathbb{Z}} a_l \, D^l : P \in \widehat{\Psi}(A) \text{ and } a_l = 0 \text{ for } l < 0 \right\} .$$

Two fundamental observations (M. Mulase):

Proposition 1. *Define:*

$$G_A = 1 + \mathcal{I}_A \ ,$$

$$\widehat{\Psi}(A)^\times = \{P \in \widehat{\Psi}(A) : P|_{t=0} \in G_A\} \ ,$$

and

$$\widehat{\mathcal{D}}_A^\times = \{P \in \widehat{\mathcal{D}}_A : P|_{t=0} = 1\} \ .$$

Then, $\widehat{\Psi}(A)^\times$ and $\widehat{\mathcal{D}}_A^\times$ are formal Lie groups: each element P in $\widehat{\Psi}(A)^\times$ or $\widehat{\mathcal{D}}_A^\times$ has an inverse of the form

$$P^{-1} = \sum_{n \geq 0} (1 - P)^n \ .$$

Recall that we write $G(*)$ for "The formal Lie group of $*$".

Proposition 2. *We have*

$$G(\Psi(A)) = \widehat{\Psi}(A)^{\times} ,$$

$$G_{+}(\mathcal{D}(A)) = \widehat{\mathcal{D}}_{A}^{\times} ,$$

and

$$G(\mathcal{I}_{A}) = G_{A} .$$

5. The amazing Mulase's factorization theorem

Theorem 4. *For any $U \in G(\Psi(A))$ there exist unique $W \in G(\mathcal{I}_A)$ and $Y \in G(\mathcal{D}_A)$ such that*

$$U = W^{-1} Y \ .$$

In other words, there exists a unique global factorization of the formal Lie group $G(\Psi(A))$ as a product group,

$$G(\Psi(A)) = G(\mathcal{I}_A) \cdot G(\mathcal{D}_A) \ .$$

This theorem allows us to solve KP as follows:

Theorem 5. *Consider the KP system of equations*

$$\frac{dL}{dt_k} = \left[(L^k)_+, L\right] \tag{7}$$

with initial condition $L(0) = L_0 \in \Psi(A)$, *and let*

$$Y \in G_+ = G_+(\mathcal{D}_A)$$

and

$$S \in G_- = G_-(\mathcal{I}_A)$$

be the unique solution to the factorization problem

$$\exp(t_k\,(L_0)^{\,k}) = S^{-1}(t_k)\,Y(t_k)\ .$$

The unique solution to Equation (7) *with* $L(0) = L_0$ *is*

$$L(t_k) = Y\,L_0\,Y^{-1}\ .$$

This is the most elementary algebraic version of the Cauchy problem for the Kadomtsev-Petviashvili hierarchy. Note that no claim on "smoothness" is made.

Now we will see a more sophisticated version using algebras equipped with valuations. One reason why this is interesting is because it is an instance of non-archimedean methods being used in "standard" mathematics.

6. KP and valuations

Let A be an associative K–algebra with unit 1, in which K is a field of characteristic zero. A is equipped with a derivation and we consider the *algebra of formal pseudo- differential operators* $P = \sum_{-\infty<\nu\leq N} a_\nu D^\nu$ over A.

We assume that A is equipped with a valuation:

Definition 1. *A valuation on an algebra A is a map $\sigma : A \to \mathbb{Z} \cup \{\infty\}$ which satisfies the following properties for all a, b in A and $k \in K$:*

- $\sigma(a) = \infty$ *if and only if* $a = 0$, *and* $\sigma(1) = 0$
- $\sigma(ab) = \sigma(a) + \sigma(b)$, *and* $\sigma(ka) = \sigma(a)$
- $\sigma(a + b) \geq min(\sigma(a), \sigma(b))$.

A valuation equips A with a topology. For $\alpha \in \mathbb{Z}$ and $x_0 \in A$ we set

$$V_\alpha(x_0) = \{x \in A : \sigma(x - x_0) > \alpha\} ;$$

the collection $\{V_\alpha(x_0)\}_{\alpha \in \mathbb{Z}}$ is a basis of neighbourhoods for a topology on A.

This topology is metrizable: we define the absolute value $|x| = c^{\sigma(x)}$ for a fixed real number $0 < c < 1$, and the metric $d(x, y) = |x - y|$.

Convergence in the metric space (A, d) is different from convergence in standard analysis! For example,

The sequence $\{a_n\}_{n \geq 0}$ is a Cauchy sequence if and only if

$$\lim_{n \to \infty} |a_{n+1} - a_n| = 0$$

.

We assume that derivation D is compatible with valuation σ:

$\sigma(D(x)) \geq \sigma(x)$ for all $x \in A$ or, $|D(a)| \leq |a|$ for all $a \in A$.

We let $\hat{A}$ be the completion of the metric space (A, d), we extend the derivation D to a continuous derivation on $\hat{A}$, and we extend the valuation σ to a valuation $\hat{\sigma} : \hat{A} \to \mathbb{R} \cup \{\infty\}$ satisfying

1. $\hat{\sigma}(x) = \sigma(x)$ for $x \in A \subseteq \hat{A}$.
2. The range of $\hat{\sigma}$ is exactly the range of σ.
3. $\hat{\sigma}$ is continuous in the topology of $\hat{A}$ induced by $|\cdot|$.

Two structures associated to the valuation $\hat{\sigma}$ on $\hat{A}$:

- The *closed* subring $\mathcal{O}_{\hat{A}} = \{a \in \hat{A} : \hat{\sigma}(a) \geq 0\}$.
- The two-sided ideal $\mathcal{P}_{\hat{A}} = \{a \in \mathcal{O}_{\hat{A}} : \hat{\sigma}(a) > 0\}$.

The derivation D on $\hat{A}$ is compatible with the valuation $\hat{\sigma}$, and so

$$D(\mathcal{P}_{\hat{A}}) \subset \mathcal{P}_{\hat{A}}\,.$$

It follows that D is well-defined on the quotient ring $\mathcal{O}_{\hat{A}}/\mathcal{P}_{\hat{A}}$. We let $\pi : \mathcal{O}_{\hat{A}} \to \mathcal{O}_{\hat{A}}/\mathcal{P}_{\hat{A}}$ be the canonical projection.

Definition 2. *The spaces of formal pseudo-differential and differential operators of infinite order are, respectively,*

$$\begin{aligned}\widehat{\Psi}(\hat{A}) = \Bigg\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \, D^\alpha : a_\alpha \in \hat{A} \text{ and } \exists A_P, B_P \in \mathbb{R}^+ \text{ and } \\ M_P, N_P, L_P \in \mathbb{Z}^+ \text{ so that } \\ M_P \geq N_P \, , \; |a_\alpha| < \frac{A_P}{\alpha - N_P} \; \forall \; \alpha > M_P \, , \; \text{ and } \\ |a_\alpha| < B_P \; \forall \; \alpha < -L_P \}\end{aligned}$$

and

$$\widehat{\mathcal{D}}_{\hat{A}} = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \, D^\alpha : P \in \widehat{\Psi}(\hat{A}) \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\} .$$

The use of the completion $\widehat{A}$ instead of A in Definition 2 is crucial in order to equip $\widehat{\Psi}(\hat{A})$ with an algebra structure.

The growth conditions in (2) are needed in the construction of a group of formal pseudo-differential operators of infinite order and of an exponential map.

The definition of $|\cdot|$ implies that $\hat{A}$ is contained in $\widehat{\Psi}(\hat{A})$.

Lemma 3.

1. *The space $\widehat{\Psi}(\hat{A})$ has an algebra structure and $\widehat{\mathcal{D}}_{\hat{A}}$ is a subalgebra of $\widehat{\Psi}(\hat{A})$.*
2. *The set $1+\mathcal{I}_{\mathcal{O}_{\hat{A}}/\mathcal{P}_{\hat{A}}}$ is a group.*

If $P = \sum_{\nu \in \mathbb{Z}} a_\nu D^\nu \in \widehat{\Psi}(\mathcal{O}_{\hat{A}})$ we set $\pi(P) := \sum_{\nu \in \mathbb{Z}} \pi(a_\nu) D^\nu$.

Definition 3. *We define the spaces*

$$G(\mathcal{O}_{\hat{A}}) = \left\{ P \in \widehat{\Psi}(\mathcal{O}_{\hat{A}}) : \pi(P) \in 1 + \mathcal{I}_{\mathcal{O}_{\hat{A}}/\mathcal{P}_{\hat{A}}} \right\}$$

and

$$G_+(\mathcal{O}_{\hat{A}}) = \left\{ P \in \widehat{\mathcal{D}}_{\mathcal{O}_{\hat{A}}} : \pi(P) = 1 \right\} .$$

Proposition 3. *The space $G(\mathcal{O}_{\hat{A}})$ is a group: each element $P \in G(\mathcal{O}_{\hat{A}})$ has an inverse of the form*

$$P^{-1} = \sum_{n \geq 0} (1 - P)^n .$$

In addition, the space $G_+(\mathcal{O}_{\hat{A}})$ is a subgroup of $G(\mathcal{O}_{\hat{A}})$.

Proposition 4. *The group* $G_-(\mathcal{O}_{\hat{A}}) = 1 + \mathcal{I}_{\mathcal{O}_{\hat{A}}}$ *is a formal Lie group with Lie algebra* $\mathcal{I}_{\mathcal{O}_{\hat{A}}}$. *The exponential map*

$$\exp : P \in \mathcal{I}_{\mathcal{O}_{\hat{A}}} \mapsto \sum_{n \in \mathbb{N}} \frac{(sP)^n}{n!} \in G_-(\mathcal{O}_{\hat{A}}) \ ,$$

in which $s \in K$, *is one-to-one, onto, and invertible.*

Theorem 6. *For any* $U \in G(\mathcal{O}_{\hat{A}})$ *there exist unique* $W \in G_-(\mathcal{O}_{\hat{A}})$ *and* $Y \in G_+(\mathcal{O}_{\hat{A}})$ *such that*

$$U = W^{-1} Y \ .$$

In other words, there exists a unique global factorization of the formal Lie group $G(\mathcal{O}_{\hat{A}})$ *as a product,* $G(\mathcal{O}_{\hat{A}}) = G_-(\mathcal{O}_{\hat{A}})\, G_+(\mathcal{O}_{\hat{A}})$.

Solution to the Cauchy problem for the KP hierarchy:

Theorem 7. *Consider the KP system of equations*

$$\frac{dL}{dt_k} = \left[(L^k)_+, L\right] \tag{8}$$

with initial condition $L_0 = \sum_{\nu\in\mathbb{Z}} a_\nu D^\nu \in \Psi(\mathcal{O}_{\hat{A}})$ *such that*
$\hat{\sigma}(a_\nu) \geq 1$ *for all* $\nu \geq 0$.

Let $Y \in G_+(\mathcal{O}_{\hat{A}})$ *and* $S \in G_-(\mathcal{O}_{\hat{A}})$ *be the unique solution to the factorization problem*

$$U = \exp(t_k\, L_0^k) = \sum_{n\geq 0} \frac{t_k^n}{n!}\,(L_0^k)^n = S^{-1}\, Y\ .$$

The unique solution to Equation (8) *with* $L(0) = L_0$ *is*

$$L(t_k) = Y\, L_0\, Y^{-1}\ .$$

7. KP in Geometry and Analysis

7.1. KP and Frölicher spaces

Definition 4. *Let X be a set.*

• *A **p-parametrization** of dimension p on X is a map from an open subset O of $\mathbb{R}^p$ to X.*

• *A **diffeology** on X is a set $\mathcal{P}$ of parametrizations on X such that, for all $p \in \mathbb{N}$,*

- any constant map $\mathbb{R}^p \to X$ is in $\mathcal{P}$;

- if $\{f_i : O_i \to X\}_{i\in I}$ is a family of compatible maps that extend to a map $f : \bigcup_{i\in I} O_i \to X$ and $\{f_i : O_i \to X\}_{i\in I} \subset \mathcal{P}$, then $f \in \mathcal{P}$.

- Let $f \in \mathcal{P}$, defined on $O \subset \mathbb{R}^p$. Let $q \in \mathbb{N}$, O' an open subset of $\mathbb{R}^q$ and g a smooth map (in the usual sense) from O' to O. Then, $f \circ g \in \mathcal{P}$.

*- If $\mathcal{P}$ is a diffeology on X, then $(X, \mathcal{P})$ is called a **diffeological space**. Let $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ be two diffeological spaces; a map $f : X \to X'$ is **smooth** if and only if $f \circ \mathcal{P} \subset \mathcal{P}'$.*

Diffeologies were introduced by Jean-Marie Souriau.

Astérisque

JEAN-MARIE SOURIAU

Un algorithme générateur de structures quantiques

Astérisque, tome S131 (1985), p. 341-399

<http://www.numdam.org/item?id=AST_1985__S131__341_0>

A recent treatise on diffeologies is

Patrick Iglesias-Zemmour, *Diffeology* AMS Mathematical Surveys and Monographs **185** (2013)

"Diffeology is an extension of differential geometry. With a minimal set of axioms, diffeology allows us to deal simply but rigorously with objects which do not fall within the usual field of differential geometry: quotients of manifolds (even non-Hausdorff), spaces of functions, groups of diffeomorphisms, etc. The category of diffeology objects is stable under standard set-theoretic operations, such as quotients, products, coproducts, subsets, limits, and colimits. With its right balance between rigor and simplicity, diffeology can be a good framework for many problems that appear in various areas of physics."

"Actually, the book lays the foundations of the main fields of differential geometry used in theoretical physics: differentiability, Cartan differential calculus, homology and cohomology, diffeological groups, fiber bundles, and connections."

Diffeologies appear in Algebraic Topology: K.T. Chen, *Iterated Path Integrals*. Bulletin AMS **83** (1977), 831-879,

and in General Relativity: C. Blohmann, M.C. Barbosa Fernandes and A. Weinstein, *Groupoid symmetry and constraints in General Relativity.* Commun. Contemp. Math. 15, 1250061 (2013) [25 pages].

We use the following subcategory of the category of diffeological spaces:

Definition 5. *A **Frölicher** space is a triple $(X, \mathcal{F}, \mathcal{C})$ in which*

- $\mathcal{C}$ is a set of paths $\mathbb{R} \to X$,

- $\mathcal{F}$ is the set of functions from X to $\mathbb{R}$

satisfying:

- A function $f : X \to \mathbb{R}$ is in $\mathcal{F}$ if and only if for any $c \in \mathcal{C}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$;

- A path $c : \mathbb{R} \to X$ is in $\mathcal{C}$ if and only if for any $f \in \mathcal{F}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.

*Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces; a map $f : X \to X'$ is **differentiable** if and only if*

$$\mathcal{F}' \circ f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R}) .$$

Smooth Manifold $\Rightarrow$ Frölicher space $\Rightarrow$ Diffeological space

Our smooth solution to the Cauchy problem for KP appears in

Well-posedness of the Kadomtsev-Petviashvili hierarchy, Mulase factorization, and Frölicher Lie groups.

Jean-Pierre Magnot and E.G.R.; Annales Henri Poincaré 21 (2020), no. 6, 1893–1945.

We build regular Frölicher Lie groups and Lie algebras of formal pseudo-differential operators in one independent variable. We actually need to adapt Mulase's constructions so as to obtain groups and algebras with Frölicher structures.

Combining these constructions with a smooth version of Mulase's algebraic factorization, we prove the well-posedness of the Cauchy problem for the Kadomtsev-Petviashvili (KP) hierarchy in a smooth category.

We can also generalize these results to a KP hierarchy modelled on formal pseudo-differential operators with coefficients which are series in formal parameters, we describe a rigorous derivation of the Hamiltonian interpretation of the KP hierarchy, and we discuss how solutions depending on formal parameters can lead to sequences of functions converging to a class of solutions of the standard KP-I equation

$$\frac{3}{4}u_{yy} - \left(u_t - \frac{1}{4}u_{xxx} - 3x_x u\right)_x = 0 \; .$$

Theorem 8. *Consider the KP system*

$$\frac{dL}{dt_k} = \left[(L^k)_+, L\right] \tag{9}$$

with initial condition $L(0) = L_0$. *Then,*

1. *There exists a pair* $(S, Y) \in G_{A_t} \times \overline{\mathcal{D}}(A_t)^\times$ *such that the unique solution to Equation* (9) *with* $L(0) = L_0$ *is*

$$L(t_1, t_2, t_3, \cdots) = Y\, L_0\, Y^{-1} = S L_0 S^{-1} .$$

2. *The pair* (S, Y) *is uniquely determined by the decomposition problem*

$$exp\left(\sum_{k\in\mathbb{N}} t_k (L_0)^k\right) = S^{-1} Y.$$

3. *The solution operator L is smoothly dependent on the variable $t = (t_1, t_2, \cdots)$ and on the initial value L_0. This means that the map*

$$(L_0, s) \in (\partial + \Psi^{-1}(R)) \times T \mapsto \sum_{n\in\mathbb{N}} \left(\sum_{|t|=n} [L(s)]_t \right) \in (\partial + \Psi^{-1}(A_t))^{\mathbb{N}}$$

is smooth, in which $s \in T = \cup_{n\in\mathbb{N}} \mathbb{K}^n$ and this set is equipped with the structure of locally convex topological space given by inductive limit.

Conclusion:

We pose the KP equations (9) on a *Frölicher algebra* $\Psi(A_t)$ and we *solve* the Cauchy problem for (9) using an *analytically rigorous factorization in Frölicher Lie groups* of an infinite-dimensional *regular Frölicher Lie group* $G(\overline{\Psi}(A_t))$.

7.2. KP and rigorous pseudo-differential operators

So far, the operators under consideration are formal pseudo-differential operators.

They are called "formal" because they cannot be understood as operators acting on smooth maps or smooth sections of vector bundles.

Any classical non-formal pseudo-differential operator generates a formal operator, but there is no canonical way to recover a non-formal operator from a formal one.

Is there a non-formal KP equation?

Yes: The non-linear equation

$$\frac{dL}{dt_k} = [(L^k)_+, L] \tag{10}$$

can be posed on groups built via groups of pseudo-differential operators.

Working with this group we can:

1. prove an analogue of the Mulase decomposition;
2. introduce a parameter-dependent KP hierarchy;
3. solve its corresponding Cauchy problem.

This report is mainly based on the following papers:

1. Mulase, M.; Solvability of the super KP equation and a generalization of the Birkhoff decomposition. *Invent. Math.* **92** (1988), 1–46.

2. Magnot, J-P.; On $Diff(M)-$pseudodifferential operators and the geometry of non linear grassmannians. *Mathematics* **4**, 1, (2016).

3. Magnot, J.P. and Reyes, E.G., Well-posedness of the Kadomtsev-Petviashvili hierarchy, Mulase factorization, and Frölicher Lie groups. *Annales Henri Poincaré* 21 (2020), no. 6, 1893–1945.

4. Eslami Rad, A.; Magnot, J.-P.; Reyes, E.G. The Cauchy problem of the Kadomtsev-Petviashvili hierarchy with arbitrary coefficient algebra. *Journal of Nonlinear Mathematical Physics* 24:sup 1, (2017), 103–120.

5. Magnot, J.P. and Reyes, E.G., $Diff_+(S^1)-$pseudo-differential operators and the Kadomtsev-Petviashvili hierarchy. arXiv 1808.03791.