Nevanlinna and algebraic hyperbolicity

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$$R_M(p,v) = \inf\left\{ \frac{1}{R} \;\middle|\; \text{there exists a holomorphic map } f : \triangle(R) \to M \right.$$
$$\left. \text{with } f(0) = p, f'(0) = v \right\}$$

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- Picard hyperbolicity. If the "$\triangle^*$-extension property" holds, then M is said to be Picard hyperbolic, ,i.e., Every holomorphic map $f : \triangle^* \to M$ extends to a holomorphic map $f : \triangle \to \overline{M}$. Kwack and Kobayashi proved that if M is Kobayashi hyperbolic and is also hyperbolically embedded in some compactification $\overline{M}$, then M is Picard hyperbolic.

Algebraic hyperbolicity

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- Ariyan Javanpeykar recently had a series of papers about the arithmetic and geometric properties for an algebraic hyperbolic X (the height inequality plays an important role).

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The First Main Theorem:

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- **Nevanlinna 1929**: Let f be meromorphic (non-constant) on $\mathbb{C}$ and $a_1, ..., a_q \in \mathbb{C} \cup \{\infty\}$ distinct. Then, for any $\delta > 0$,

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- **Siu-Yeung, 1997** (Noguchi-W-Y). Let A be an abelian variety and D be an ample divisor on A. Let $f : \mathbb{C} \to A$ be holomorphic with $f(\mathbb{C}) \not\subset D$. Then

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- $\log \sigma$ is harmonic outside possibly a finite set $\Sigma := \{P_1, \ldots, P_k\}$ on Y.
- At each $P_i \in \Sigma$, in a coordinate chart (U, z) centered at P_i that does not contain other points in Σ, $\log \sigma(z) = k_i \log |z| + h_{P_i}(z)$, where h_{P_i} is a harmonic function on U.

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Let $B(r) := \{y \in Y : \sigma(y) < r\}$ and $S(r) := \{y \in Y : \sigma(y) = r\}$. Let $d\mu_r = d^c \log \sigma|_{S(r)}$. Let $\varsigma := \int_{S(r)} d\mu_r$, which is is independent of r for r large enough.

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where, for a positive real number x, $\log^+ x = \max\{0, \log x\}$ and $\log^- x = -\min\{0, \log x\}$. We say that (X, D) is Nevanlinna hyperbolic if there is a positive $(1,1)$-form η on X such that for any parabolic Riemann surface Y and every holomorphic map $f : Y \to X$ with $f(Y) \not\subset D$ and for $\delta > 0$, one has

$$T_{f,\eta}(r) \leq_{\text{exc}} \overline{N}_f(r, D) - \mathfrak{X}_\sigma(r) + (\delta + 2\varsigma)\log r + \mathfrak{E}_\sigma(r) + O(1).$$

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$T_{f,\eta}(r) \leq_{\text{exc}} \overline{N}_f(r, D) - \mathfrak{X}_\sigma(r) + (\delta + g + 1)\log r + O(1)$. Note $\mathfrak{X}_\sigma(r) = \int_1^r \chi_\sigma(t)\frac{dt}{t}$, where $\chi_\sigma(t)$ is the Euler characteristic of the domain $B(t)$.

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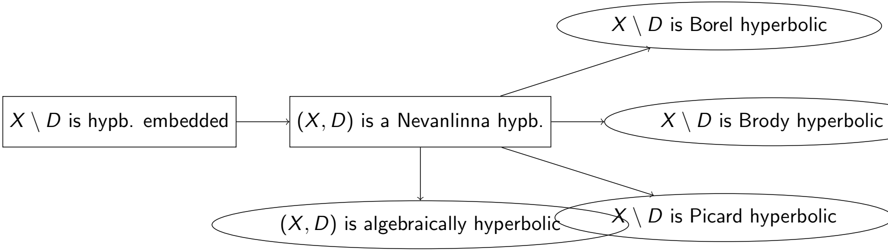

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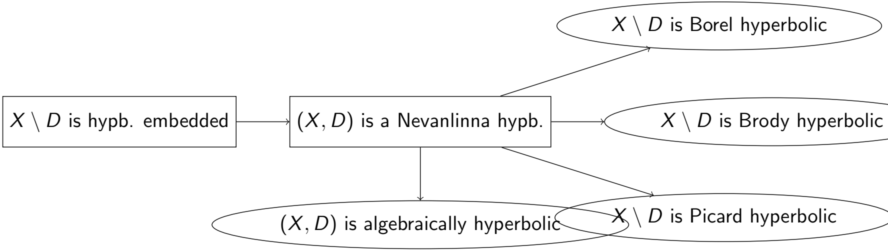

So the notion of Nevanlinna hyperbolicity links and unifies the Nevanlinna theory, the complex hyperbolicity (Brody and Kobayashi hyperbolicity), the big Picard type extension theorem (more generally the Borel hyperbolicity), as well as the algebraic hyperbolicity.

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Thus, $T_{f,\omega}(r) = \int_{r_1}^{r}\left(\int_{r_1\leq|z|\leq\rho} f^*\omega\right)\frac{d\rho}{\rho} \leq C\log r$.

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But, by the standard Nevanlinna theory trick (Green-Jensen and calculus lemma), we can get (logarithmic derivative lemma)

$$T_{\mathrm{Ric}[\omega_{Y*}]}(r) \leq_{exc} (1+\delta)^2 \log T_{\omega_{Y*}}(r) - \mathfrak{X}_\sigma(r) + (\delta + 2\varsigma) \log r + O(1).$$

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This derives our desired inequality.

Urata's type theorem:

Theorem. Let $\bar{X}$ be a smooth projective variety over $\mathbb{C}$ and let $D \subset \bar{X}$ be a divisor such that $(\bar{X}, D)$ is Nevanlinna hyperbolic. If C is a smooth quasi- projective connected curve over $\mathbb{C}$ with smooth projective model $\bar{C}$, $c \in \bar{C}(\mathbb{C})$, and $x \in \bar{X}(\mathbb{C})$, then the set of morphisms $\bar{f} : \bar{C} \to \bar{X}$ with $\bar{f}(C) \subset X$ and $\bar{f}(c) = x$ is finite.

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Theorem. Let V be a Cohen–Macaulay complex projective variety of dimension n. Let $D_0, D_1, ..., D_r$, $r \geq n+1$, be effective Cartier divisors of V in general position. Suppose that there exist an ample Cartier divisor A on V and positive integers d_i such that $D_i \equiv d_i A$ and $d_i \geq d_0$ for all $0 \leq i \leq r$. Let $f : \mathbb{C} \to X$ be a holomorphic map. Assume that the following
(i) $r > (n+1)^2$ and $\frac{1}{d_i} f^* D_i \leq \frac{1}{d_0} f^* D_0 + O(1)$ for all $i = 0, \ldots, r$;
or
(ii) $r > n^2 + n + 1$ and $\sum_{i=1}^{r} \frac{1}{d_i} f^* D_i \leq \frac{1}{d_0} f^* D_0 + O(1)$.
Then f is constant.

As a special case of the above Theorem, we get

Theorem. Let $n \geq 2$, $F_1, ..., F_r, G \in \mathbb{C}[X_1, ..., X_n]$ be polynomials in general position (i.e. the associated hypersurfaces are in general position) with $\deg(F_i) \geq \deg(G)$ for $i = 1, \ldots, r$. Let $h_1, ..., h_n$ be holomorphic functions on $\mathbb{C}$ such that one of the following holds

(i) $r > \frac{n(n+3)}{2}$ and $\frac{G(h_1,...,h_n)}{F_i(h_1,...,h_n)}$ is holomorphic, for $i = 1, ..., r$; or

(ii) $r > \frac{n^2+n+2}{2}$ and $\frac{G(h_1,...,h_n)}{\prod_{i=1}^r F_i(h_1,...,h_n)}$ is holomorphic.

Then $[h_1 : \cdots : h_n]$ is constant.